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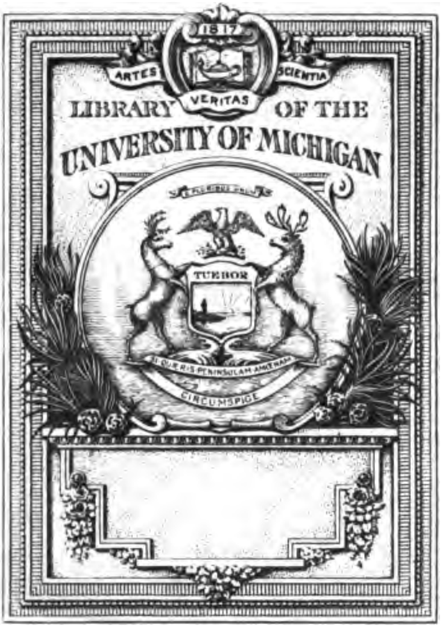
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THE NEW

Mathematician's Guide:

CONTAINING THE

ELEMENTS

OF

UNIVERSAL MATHEMATICS,

And demonſtrating

Sir *ISAAC NEWTON*'s

METHOD of finding DIVISORS.

WITH

RULES for extracting the Root of a BINOMINAL,

And for determining the FORM of an

ASSUMED INFINITE SERIES.

Written Originally in *Latin*,

Willem Jacob van 's

By the Learned Dr. *GRAVESANDE*.

LONDON:

Printed for S. AUSTEN in *Newgate-Street*.
MDCCXLIX.

Hist. of Science
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PREFACE.

AMongſt the various Uſes of the Mathematics, their rendring the Underſtanding more ſuſceptible of other Sciences, is certainly the moſt excellent. And this it does in a twofold manner: Firſt, in accuſtoming the Mind to reaſon juſtly, by which there is acquired a Faculty of diſcovering the Defects and Fallacies of falſe Reaſoning; and of ſeeing what is neceſſary to make it concluſive: And Secondly, the Mind by the Study of the Mathematics attains ſuch a Sagacity in ſeeking Truth, as is of great Importance, not only in other Sciences, but even in the Affairs of Life if the Knowledge of other Sciences and of Men, is join'd to the Mathematics. For he that, whilſt he applies himſelf to the Mathematics, neglects every thing elſe, does not properly learn a Mathematic Method, but only makes himſelf readier at reaſoning about Quantities.

The firſt of theſe Uſes is beſt promoted by the Study of Geometry, if we follow the Antients Method of Demonſtrating; for it is not learning a great deal in a little time, but learning a little well, that ought to be regarded: ſince in acquiring a Faculty of reaſoning well, we ought not to conſider the number of Propoſitions, ſo much as the Method they are deliver'd in; and this is what latter Geometers have not ſufficiently attended to.

12-11-39 HCM

The ſecond Uſe we have mentioned, is by many, deſervedly referred to Algebra, which gives Rules and Examples for the diſcovery of Truth : wherefore the Study of this Science ought to be join'd to the Study of Geometry.

And he that would compleat himſelf in the Art of Reaſoning juſtly, ſhould join to both, the Study of Logic.

If to this Uſe of Algebra, we add the Neceſſity of it in all parts of the Mathematics, but eſpecially in Phyſics, which none can attempt without the Elements of Geometry and Algebra; we need ſay no more in the praiſe of it.

The Elements that I here deliver, were firſt intended for private uſe; and I have therefore been as brief as poſſible.

I do not here treat of Problems of more than two Dimenſions, my chief intent being to explain as much as poſſible, the nature of Problems, which the greateſt part of the Writers of Algebra neglect, who are often defective too in deducing the Reaſons of the Operations from the general Rules.

For this cauſe I have added many Obſervations relating to the Solution of Problems; and have treated amply of Problems of two Dimenſions.

I muſt have ſaid much more if I had treated of Problems of three and four Dimenſions. Many have treated of the Nature of Equations; but the Nature of thoſe Problems that produce theſe Equations, has been too much neglected. It is nevertheleſs a Matter worthy a Mathematician, and which will, when explain'd, free young Beginners from many Difficulties.

As to the two little Treatiſes added to theſe Elements, ſee their reſpective Prefaces.

THE ELEMENTS OF Univerſal Mathematics.

CHAP. I.

Of Quantity in general, and the Intention of Algebra.

THE general Object of the Mathematics is Quantity.

1. *Every thing that can be increaſed or diminiſhed, we ſay has Quantity:* as Extenſion, Time, Motion, Velocity, *&c.*

In different Parts of the Mathematics, different ſorts of Quantity are examin'd; but I ſhall here treat of Quantity conſider'd in general, and briefly expound the Rules by which Truth is diſcover'd, when Quantities of any kind whatever are treated of.

Thoſe things which are demonſtrated concerning Quantity in general, may be applied to any peculiar Quantity. And we attain the Truth by the ſame method in any Parts of the Mathematics.

2. *All Reaſoning concerning Quantity may be referred to its Meaſure.*

3. *The Meaſure of any Quantity is the comparing it with another of the ſame kind.* A Line is ſaid to be of ſix feet, if being compar'd with a Line of one foot, it contains it ſix times.

Quantities of the ſame kind only admit of Compariſon; a Line with a Line, a Superficies with a Superficies, and Time may be compared with Time: But no Time can ever be ſaid to be equal to any Line whatſoever.

4. When we compare one Quantity with another, we often regard the Exceſs of one above the other; that is, we ſee the greater to be equivalent to the Sum of the leſſer with ſome other: and this we may apply to many Quantities, the Compariſon ſtill repeated. To this way of comparing may be referred that Operation of Arithmetic, which we call ADDITION.

5. Oftentimes only changing the Conſideration, we regard the Defect by which one Quantity falls ſhort of another. This determining the difference of two Quantitys is call'd SUBTRACTION.

6. A Quantity which is form'd by the repeated Addition of the ſame Quantity, is ſaid to be form'd by MULTIPLICATION.

7. The contrary Operation, by which we determine how often one Quantity is contain'd in another, is call'd DIVISION.

8. *No Compariſon of Quantitys can be conceiv'd, which may not be referred to one of theſe four Operations.* For as to the Extraction of Roots, it will be ſhown in its proper time, that it ought to be referred to Diviſion.

9. By theſe four Operations every Truth, that relates to Quantities, is inveſtigated.

I will now relate what I intend to explain in theſe Elements, and in what Order.

In the firſt place I will briefly ſhow how Quantity may be ſo generally expreſſed, that the Expreſſion may be applied to any particular Quantity.

Afterwards I will explain how the Operations relating to Quantities ſo expreſſed, are perform'd.

But as new Expreſſions may ariſe from theſe Operations; I will demonſtrate by what method theſe new Expreſſions are to be managed.

Laſtly, I will tell by what Art and in what Order theſe Operations are to be directed in the diſcovery of Truth.

10. This Art is called ALGEBRA, as alſo ANALYSIS, and by ſome UNIVERSAL MATHEMATICS.

The Scope of this Art appears from what has been ſaid to be the Solution of Problems.

11. Mathematicians call every Propoſition a *Problem*, which requires any thing to be done.

But that a Problem may be ſolv'd, that which is deſir'd is firſt to be perform'd; and then it is to be demonſtrated that the Solution ſatisfies the Conditions ſought.

In Analytic Solutions the Operations by which they are attain'd, put it out of doubt that the Solutions ſo diſcover'd are true.

CHAP. II.

Of the Expreſſions of Quantities, and the Operations about ſimple Quantities.

12. Mathematicians have always thought it very *commodious to expreſs Quantities by Letters* in Algebraic Operations.

Who does not ſee that the ſingle Letters *a*, *b*, *c*, &c. may expreſs any particular Quantity whatſoever, either known or unknown? And that the unknown are ſubject to the ſame Operations which take place in known Quantities.

13. But that known Quantities may be more eaſily diſtinguiſh'd from unknown, theſe are denoted by the laſt Letters *z*, *y*, *x*, *u*; whilſt the firſt, as *a*, *b*, *c*, ſerve to deſign known Quantitys.

Of ADDITION.

14. *The Sum of ſeveral Quantities is expreſſed by joining the Letters that denote each ſingle Quantity with this Sign* +, *which ſignifies* PLUS, or *more.* So $a + b$, is *a* plus *b*, and is equivalent to the Sum of the Quantitys *a* and *b*.

EXAMPLE.

			$3d$
	a	d	$2d$
a	b	f	$2g$
a	a	g	g
$2a$	$2a+b$	$d+f+g$	$5d+3g$

Of

OF SUBTRACTION.

15. This Sign — ſhews the Quantity to be taken from another, and ſignifies MINUS, or *leſs.* $a - b$ is a minus b, that is, it expreſſes the Difference of the Quantities a and b.

EXAMPLES.

$$\begin{array}{ccc} 3a & a & 3a \\ 2a & b & 2b \\ \hline a & a - b & 3a - 2b \end{array}$$

OF MULTIPLICATION.

Multiplication is a Repetition of the ſame Quantity or of its Part (6), ſo that the Number of Repetitions are to be expreſſed, and theſe are always expreſſed by Numbers.

16. *Every Multiplication* therefore *is perform'd by Number, and to multiply Quantity by Quantity is abſurd.*

17. But *Quantity*, if compared with another which is accounted for Unity, *may* be deſign'd by a Number, and be itſelf *accounted for a Number, notwithſtanding it is expreſſed by a Letter*; which is always done when one Quantity is multiplied by another, to avoid Abſurdity: neither is the Determination of Unity in this caſe neceſſary.

18. *The Sign of Multiplication is this* ×, *or a ſimple joining of the Letters.* When a is to be multiplied by b, we write $a \times b$, or $a\,b$.

19. *This* ($a\,b$) *is called the* PRODUCT *of the Multiplication.*

20. Which ſame Product may be expreſſed by ba; for the Order of the Letters is not regarded in Multiplication.

If $3a$ is to be multiplied by b, the Product is $3ab$; but if $3a$ is to be multiplied by $2b$, it is evident the former Product muſt be doubled, and is $6ab$. Therefore,

21. *When the Quantities are multiplied by Numbers, let theſe Numbers be multiplied ſeparately.*

22. The Product of a by a is aa, which is alſo thus expreſs'd a^2.

If aa or a^2 is multiplied by a, the Product is aaa or a^3.

23. For this reaſon, *when a Quantity is multiplied by a Number,* that all Confuſion may be avoided, *let that Number be written before the Letter.* If b is multiplied by 6, that is, if it ought to be taken ſix times; I write it $6b$ not $b6$ leſt this Expreſſion be confounded with b^6, which ſignifies $bbbbbb$.

24. The Product of a by a, or aa, is called THE SQUARE OR SECOND POWER of the Quantity a; and a is the ſquare ROOT of aa.

25. The *Cube* of the Quantity a is a^3 or aaa, it is alſo called the THIRD POWER; and a is the Cube Root of a^3.

26. The FOURTH POWER of the Quantity a, is a^4, which is alſo called the SQUARED SQUARE, THE FIFTH POWER is a^5, and ſo of the reſt.

27. *The Number annex'd to* a, *is called the* INDEX *of its Power.*

28. *The Quantity which is to be multiplied by another, is called the* MULTIPLICAND.

29. *The* MULTIPLICATOR *is that Quantity by which the Multiplication is perform'd.*

30. *That Quantity is called the* PRODUCT *of the Multiplication, which ariſes from it, as has been obſerv'd in* n. 19.

EXAMPLES OF MULTIPLICATION.

$$\begin{array}{ccccc} a & a^3 & ab & 3b & 3a^2f \\ b & ab & ab & 6d & 4af \\ \hline ab & a^4b & a^2b^2 & 18bd & 12a^3f^2 \end{array}$$

OF DIVISION.

31. *Diviſion reſolves what was compounded by Multiplication*, determining how often one Quantity is contain'd in another.

32. *The Quantity which is to be divided, is called* THE DIVIDEND.

33. *The* DIVISOR *is the Quantity by which the Diviſion is perform'd.*

34. *The* QUOTIENT *is the Quantity which expreſſes how often the Diviſor is contained in the Dividend.*

35. *It is ſufficiently manifeſt that the Product ariſing from the Diviſor multiplied into the Quotient, is equal, or the ſame with the Dividend.*

36. Whence we diſcover this Rule for Diviſion. *From the Dividend blot out the Diviſor, and the remainder is the Quotient:* If *a b* is to be divided by *a*, blotting out *a*, there will remain *b*, which is the Quotient; for this Quotient *b* multiplied by the Diviſor *a*, produces in the Product the Dividend *a b*.

37. *If there are Numbers to the Letters*, it is evident *they ought to be ſeparately divided* (21. 35.)

EXAMPLES of DIVISION.

$$\left.\begin{array}{l}\text{Divid. } ab\\ \text{Divis. } \quad a\end{array}\right\}b \text{ Quot.} \left.\begin{array}{c}a^3bd\\ abd\end{array}\right\}aa. \left.\begin{array}{c}6afg\\ 3ag\end{array}\right\}2f.$$

38. *When Diviſion cannot be performed by the method,* as if *ab* is to be divided by *f*, *the Dividen is to be written above the Line, and the Diviſo under the ſame;* ſo $\frac{ab}{f}$ expreſſes the Quotient of this Diviſion.

39. *The Quotient in this laſt caſe is call'd a* FRACTION.

40. *The Dividend* (as *ab*) *is called the* NUMERATOR *of the Fraction.*

41. *And the Diviſor* (*f*) *is the* DENOMINATOR *of the Fraction.*

CHAP. III.

Of Operations about compound Quantities.

42. A Quantity is ſaid to be compound, when it conſiſts of various ſimple ones join'd with the Signs + or —.

OF ADDITION.

43. It is to be obſerv'd that a Quantity written without any Sign, is underſtood to have the Sign + annex'd; and it is evident that *Addition is perform'd by joining the Quantities to be added, preſerving their Signs.*

EXAMPLES.

$$
\begin{array}{lll}
a+b & 2a-f & 2a+g-d \\
d+f & a-l & 3f-g-a \\
 & & b-g+d \\
\hline
a+b+d+f & 3a-f-l & a+3f+l-g
\end{array}
$$

OF SUBTRACTION.

44. Subtraction is the contrary to Addition; therefore *the Quantity to be ſubtracted muſt have all its Signs chang'd, and then be added to the other.*

EXAMPLES.

$$\begin{array}{ccc} a+b & 2a+b & 3a+f-2g \\ c+d & a-b & 4a+g-d \\ \hline a+b-c-d & a+2b & f-3g-a+d \end{array}$$

OF MULTIPLICATION.

45. It will be evident, if attentively conſider'd, that *each Quantity of the Multiplicand ought to be multiplied ſeparately by each of the Quantities in the Multiplicator.*

46. In each of theſe particular Multiplications it is to be obſerv'd, *if the Signs of the Multiplicand and Multiplicator are ſimilar (or the ſame) the Product will be Affirmative, but Negative if they are different.*

47. This Rule may be deduc'd from the Examination of all the particular Caſes, which it is plain can be but four, for every Multiplication is perform'd by a Number 16): Therefore, 1. An affirmative or poſitive Quantity may be multiplied by an affirmative Number. 2. An affirmative Quantity by a negative Number. 3. A negative Quantity by an affirmative Number. 4. Laſtly, a negative Quantity by a negative Number.

In the firſt Caſe, who can deny the Product to be affirmative?

In the ſecond Caſe, it is negative; for to mu tiply by a Negative is to take away, ſince to mu tiply by an Affirmative is to add.

In the third Caſe, there is a negative Quan tity which is taken certain times, and therefor remains negative.

The fourth Caſe is contrary to the third therefore the Product muſt have a contrary Sign, or muſt be affirmative. There is taken away in this Caſe a negative Quantity, by which the Negation vaniſhes. So to take away a Debt is to pay it.

EXAMPLE I.

$$\begin{array}{l} a+b \\ d+f \\ \hline ad+bd+af+bf \end{array}$$

EXAMPLE II.

$$\begin{array}{llllllll} 3a+2f-g-m \\ 2a+g-m \\ \hline 6aa+4af & -2ag & -2am & +2gf & -gg & -gm & -2fm & +m^2 \\ & +3ag & -3am & & & +gm & & \\ \hline 6aa+4af & +ag & -5am & +2gf & -gg & & -2fm & +m^2 \end{array}$$

48. We have ſaid (18), that Multiplication may be expreſs'd by this Sign ×; this often happens in compound Quantities, in which caſe Lines are drawn above the Quantities to be multiplied, which ſeparate them from the reſt, thus $\overline{a+b}\times\overline{f+g}-gf$, ſignifies gf is to be ſubtracted from the Product of the Quantities $a+b$ multiplied by $f+g$.

49. What has been ſaid of the Method of expreſſing the Powers of ſimple Quantities (21), may alſo be applied to compound Quantities, if a Line is drawn above them, as $\overline{a+b}^3$ ſignifies the Cube of the Root $a+b$.

OF DIVISION.

50. *A compound Quantity is ſaid to be diſpoſed or order'd according to the Dimenſions of a given Letter, when in the firſt place there is put a ſimple Quantity, which contains the higheſt Power of the given Letter, and the Powers ſucceſſively decreaſing are diſpoſed in the places following;*

$$ab^3 + fgb^2 + a^3b + a^4.$$

This Quantity is order'd according to the Dimenſions of the Letter b.

51. In this caſe ab^3 is the firſt Term, and the decreaſing Powers of b determine the following Terms: the number of Terms always exceeds by Unity, the Index of the higheſt Power; and this Quantity contains four Terms.

52. Oft-times ſome Terms are wanting, whoſe places muſt then be left void. In the forementioned Quantity, had it been ordered according to the Letter a, the third Term would have been wanting,

$$a^4 + ba^3 * + b^3a + fgb^2.$$

53. Sometimes particular Terms conſiſt of compound Quantities, as in this Quantity.

$$\begin{array}{lllll} e^4 & + fe^3 & + llee & * & - fl^3 \\ & - ge^3 & & & - hmst \end{array}$$

The ſecond Term is $fe^3 - ge^3$, and the fifth is $fl^3 - hmst$.

54. In theſe caſes we call that the firſt Quantity of the Term that is uppermoſt, but they may be placed at pleaſure.

Theſe things firſt remark'd, Diviſion is t
perform'd by the following Rules.

55. 1. *The Dividend and Diviſor are to be or*
according to the Dimenſions of the ſame Letter.

2. *Let the firſt Quantity of the firſt Term of*
Dividend, be divided by the firſt Quantity of
firſt Term of the Diviſor, and note the Quotient.

3. *Let the whole Diviſor be multiplied by the w*
Quotient, and let the Product be ſubtracted from
Dividend.

4. *The Diviſion will proceed by repeating t*
Operations, and the Sum of the particular Quotie
will give the Quotient ſought.

5. *In the particular Diviſions it is to be obſerv*
that ſimilar Signs of the Dividend and Diviſor gi
an affirmative Quotient, and different Signs give
Negative.

If (the Diviſion being ended) we attend to t
Operations, it will be evident that the Quotie
ariſing by theſe means is a true one, fro
Theor. n. 35.

EXAMPLE I.

$$\text{Divid.}\left\{\begin{array}{l} ab + dc \\ ac + db \end{array}\right. \left|\begin{array}{l} a + d\ \text{Divis.} \\ \hline b + c\ \text{Quot.} \end{array}\right.$$

$$\begin{array}{r} \hline -ab - db \\ \hline ac + dc \\ -ac - dc \\ \hline 0 \quad 0 \end{array}$$

Ex

EXAMPLE II.

$$\begin{array}{l|l}
\text{Divid. } a^3 + ba^2 + b^2a + b^3 & a + b \text{ Divis.} \\
\hline
- a^3 - ba^2 & aa + bb. \text{ Quot.} \\
\hline
\quad + bba + b^3 & \\
\quad - bba - b^3. & \\
\hline
\quad\quad 0 \quad 0 &
\end{array}$$

EXAMPLE III.

$$\begin{array}{l|l}
\text{Divid.} a^4 \quad * \quad * \quad * \quad - b^4 & a - b \text{ Divis.} \\
\hline
- a^4 + ba^3 & a^3 + baa + b^2a + b^3 \\
\hline
\quad + ba^3 \qquad - b^4 & \\
\quad - ba^3 + b^2a^2 & \\
\hline
\quad\quad + b^2a^2 \quad - b^4 & \\
\quad\quad - b^2a^2 + b^3a & \\
\hline
\quad\quad\quad + b^3a - b^4 & \\
\quad\quad\quad - b^3a + b^4 & \\
\hline
\quad\quad\quad\quad 0 \quad 0 &
\end{array}$$

56. When theſe Operations cannot be produced farther, and yet ſome Quantities are left in the Dividend, then the Diviſion cannot be exactly perform'd; and thoſe Quantities left are called the REMAINDER of the Diviſion.

57. In theſe caſes the whole Quotient may be deſign'd by a Fraction, or a Fraction form'd out of the Remainder may be added to the Quotient found before.

If $aa+2ab+2bb$ is to be divided by $a+b$ the Quotient will be $\frac{aa+2ab+2bb}{a+b}$, or $a+b+\frac{bb}{a+b}$

Of the EXTRACTION *of* ROOTS.

58. *The Inveſtigation of the Roots of any given Powers, is called the* EXTRACTION *of* ROOTS, and is a peculiar Species of Diviſion, in which both the Diviſor and Quotient are ſought together, their relation being given.

59. When the Queſtion is only of ſimple Quantities, the Root, if it can be extracted, is diſcovered without difficulty; a^2 has the ſquare Root a; a^4 has its Root aa; and ab is the ſquare Root of $aabb$.

In the ſame manner it is evident that a^3 has a cubic Root a, and a^3b^3 the Root ab.

I will explain what relates to the Square and Cubic Roots of compound Quantities, from whence the Extraction of the Roots of the reſt of the Powers will be eaſily deduc'd.

In the Extraction of ſquare Roots, it is to be obſerv'd, that in every Square of a compound Quantity, there is given the Squares of the particular Quantities of which the Root conſiſts, from which Obſervation we deduce the Method of diſcovering the Root to be tried.

EXAMPLE I.

Let there be ſought the ſquare Root of the Quantity $aa+2ab+bb$.

This Quantity contains two particular Squares aa and bb. I extract the Roots of theſe a and b, which I collect into one Sum, and I have $a+b$

for the Root to be tried, whoſe Square is the Quantity propos'd.

60. When the Square of the Root diſcover'd, differs from the Quantity propos'd, the Root of it cannot be extracted.

EXAMPLE II.

$$aa + ab + bb.$$

The Root to be tried is again $a + b$, whoſe Square $aa + 2ab + bb$ differs from the Quantity propos'd, and therefore the Root of it cannot be extracted.

61. How the Signs of the radical Quantities are determin'd when all the Signs of the Quantities in the Square propos'd are not affirmative, will be explain'd in the following Example.

EXAMPLE III.

$$aa + 2ab - 2bc + bb + cc - 2ac.$$

I extract the Roots of the three Squares aa, bb, cc, which are a, b, c, and as every Square is affirmative (46), the Signs as yet are unknown; but the affirmative Product $2ab$, ſhews that the Signs of the Quantities a and b are ſimilar (46); and $-2bc$ ſhews that the Signs of b and c are different: The Root to be tried is therefore $a+b-c$, or $c - a - b$, the Square of either of which is the Quantity propos'd.

62. A Square is often to be ſupplied as in the following Example.

EXAMPLE IV.

$$a^4 - 2a^3b - aabb + 2ab^3 + b^4.$$

This Quantity contains only two Squares a^4 and b^4, whoſe Roots do not ſuffice; but I ſee that a Square may have been taken away by a Quantity with a contrary Sign; for there is given $-aabb$, which Quantity would be a Square if it was affirmative. I add therefore to the Quantity propos'd $+aabb-aabb$, by which the Value of the Quantity is not chang'd. It then becomes $a^4-2a^3b-2aabb+aabb+2ab^3+b^4$, whoſe Root to be tried is $aa-ab-bb$, or $bb+ab-aa$; the Square of either of which is the Quantity propos'd.

EXAMPLE V.

$$a^4+2a^3b+3aabb+2ab^3+b^4.$$

This contains only two Squares, whoſe Roots do not ſuffice, but $3aabb$ may be expreſſed $2aabb+aabb$; in which caſe there is given a third Square $aabb$, and the Root to be tried is $aa+ab+bb$, which is the true one.

63. Cubic Roots are inveſtigated by a like Method; for from the formation of a Cube, it is certain that there is given in the Cube of a compound Quantity, the Cubes of the particular Quantities that are contained in the Root.

But as to the Signs, theſe are more eaſily diſcover'd here, becauſe a Cube is always affected with the ſame Sign as its Root, which holds true in every Power whoſe Index is an odd number.

EXAMPLE.

$$a^3-3a^2b+3abb-b^3.$$

The Root to be tried is $a-b$, whoſe Cube is the Quantity propos'd.

64. *When the Root cannot be extracted it is denoted by a particular Sign.* $\sqrt{ab}$ ſignifies the

ſquare Root of the Quantity ab; $\sqrt{aa+bb}$ is the ſquare Root of $aa+bb$.

The Sign of the cubic Root is $\sqrt[3]{}$; of the Root of the fourth Power $\sqrt[4]{}$; and ſo of the reſt.

65. *The Roots expreſſed by theſe Signs are called ſurd Quantities.*

CHAP. IV.

Of Fractions.

66. THIS general Theorem is firſt to be laid down, *That the Quotient of the Diviſion is not changed, if the Dividend and Diviſor are both multiplied or divided by the ſame Number.*

ab may repreſent any Dividend whatſoever, and a any Diviſor, then the Quotient will be b; if ab and a are each multiplied by d, then abd divided by ad, the Quotient is alſo b. The Demonſtration is the ſame with reſpect to Diviſion.

67. In every Fraction the Numerator is the Dividend, the Denominator is the Diviſor, and the Fraction is the Quotient, (38, *& ſeq.*) *therefore if the Numerator and Denominator are each multiplied or divided by the ſame Quantity, the Fraction is not changed* (66)

68. A Quantity is not changed if multiplied by Unity, and therefore neither is it, if divided by Unity (31); conſequently,

69. *We may change a whole Quantity into a Fraction, if whilſt the Quantity it ſelf is placed for the Numerator, it has Unity for a Denominator.*

$a+b$ is the ſame with $\frac{a+b}{1}$.

70. *A Whole Quantity is reduced into a Fracti whoſe Denominator is given,* if the Numerator a Denominator of the Fraction diſcover'd by t foregoing Rule, are each multiplied by the give Denominator (67). So $a+b$ is reduced to Fraction whoſe Denominator is d, if we chang the Fraction $\frac{a+b}{1}$ into $\frac{ad+bd}{d}$.

71. Which Rule may be applied to Unity its ſelf; and $\frac{1}{1}$ is equivalent to Unity, as is alſo $\frac{d}{d}$ and $\frac{aa+bb}{aa+bb}$.

A Fraction may be reduced to another more ſimple, if the Numerator and Denominator have a common Diviſor (67). And by how much greater this Diviſor is, by ſo much will the Fraction, when reduced, be the ſimpler.

72. *A Fraction is reduced to the ſimpleſt poſſible, if the Numerator and Denominator are divided by their greateſt common Diviſor.*

I will for this purpoſe lay down a Method by which the greateſt common Diviſor of two Quantities may be found.

OF FINDING THE COMMON DIVISOR.

Plate I. Fig. I. 73. Let AB and CD be given Lines, whoſe greateſt common Meaſure is ſought.

If the leſs exactly meaſures the greater, then will the leſſer CD be the Meaſure ſought.

But if after the leſſer has been taken from the greater as often as poſſible, there is ſtill a Remainder, as EB; it is evident that the Meaſure ſought that meaſures CD, will alſo meaſure it, how often ſoever repeated: that is, it will meaſure A E; but it ought alſo to meaſure A B, therefore E B likewiſe.

We are therefore to ſeek the greateſt common Meaſure of the Remainder E B, and the leſſer C D.

If the Remainder E B exactly meaſures C D, E B will be the Meaſure ſought. If it does not meaſure it, let it be taken from C D as often as it can, then by a Demonſtration ſimilar to the precedent, it will be found that we muſt ſeek the common Meaſure of this laſt Remainder, and the precedent one E B; and the Operation is in the ſame manner to be continued, till the laſt Remainder exactly meaſures the precedent Remainder. And from hence we deduce this Rule:

74. *Two Quantities being given, whoſe greateſt common Meaſure is ſought; divide the greater by the leſſer, rejecting the Quotient; divide the leſſer by the Remainder, and continue the Operation, always rejecting the Quotients, and dividing the Diviſor of the laſt Diviſion by the Remainder of the ſame till you come to a Diviſion without a Remainder, the Diviſor of this Diviſion is the Quantity ſought.*

This Rule is ſufficient whilſt Numbers only are conſider'd, and brings to Unity, if they have no other common Meaſure.

If the Quantities propos'd are Algebraic, there are theſe two things beſides to be obſerv'd in each Diviſion.

75. *Try whether the whole Diviſor can be divided by any Quantity either ſimple or compound, by which the firſt Term of this Diviſor may be divided.*

76. *If the firſt Term of the Diviſor does not exactly divide the firſt Term of the Dividend, the whole Dividend is to be multiplied by a Quantity which gives this Diviſion exactly.* This Quantity will be

diſcover'd by a ſimple Conſideration of the Qu tities, with a ſmall Attention.

EXAMPLE I.

Let there be ſought the greateſt common I viſor of theſe two Quantities, each of which have order'd according to the Dimenſions of t Letter *a*.

$$\begin{array}{llll} a^3 & * & +bda & +bbd \\ & & -bba & -bdd \\ & & -dda & \end{array}$$

And

$$a^3 + daa - bba - bbd.$$

1ſt DIVISION.

I divide the firſt by the ſecond, and the Quotient is 1, which I neglect; the Remainder is

$$\begin{array}{lll} -daa & +bda & +2bbd \\ & -dda & -bdd, \end{array}$$

which I divide by *d* (75), and I have

$$\begin{array}{lll} aa & +ba & +2bb \\ & -da & -bd. \end{array}$$

2d DIVISION.

By this Quantity I divide the firſt Diviſor, neglecting the Quotient $-a$, and there remains

$$\begin{array}{lll} baa & +bba & -bbd \\ & -bda, & \end{array}$$

which divided by *b* (75), produces

$$\begin{array}{lll} aa & +ba & -bd \\ & -da. & \end{array}$$

3d DIVISION.

By this laſt Remainder I divide the Diviſor of the laſt Diviſion, the Quotient is -1, and the Remainder is

2ba

$$\begin{array}{r} 2ba + 2bb \\ -2da - 2bd. \end{array}$$

Which Quantity I divide by $2b - 2d$, and $a + b$ the Quotient is the Diviſor of the fourth Diviſion (75).

4th Division.

By $a + b$ I divide the Diviſor of the third Diviſion, and there is no Remainder; therefore $a + b$ is the greateſt common Diviſor ſought.

Example II.

There are Quantities given, order'd according to the Dimenſions of the Letter a.

$$\begin{array}{l} aaff + abdg - ddgg \\ aafh. \end{array}$$

And

$$\begin{array}{l} aahh + afdg - ddgg \\ aafh. \end{array}$$

The greateſt common Diviſor of theſe is ſought.

1ſt Division.

I divide the firſt Term of the firſt Quantity by the firſt Term of the ſecond, the Quotient is $\frac{f}{h}$; I multiply therefore the firſt Quantity by h, to take away the Fraction (76), and then the Quotient is f; the Remainder is

$$\begin{array}{r} ahhdg - ddggh \\ -affdg + ddggf \end{array}$$

The whole Quantity being divided by dg, there is produced

$$\begin{array}{r} ahh - dgh \\ -aff + dgf \end{array}$$

Both whoſe Terms may be divided by $b-f$, ſo that it is reduced to this

$$ab+af-dg.$$

Which exactly divides the Diviſor of the firſt Diviſion, and is the common Diviſor ſought.

77. It is to be obſerv'd, that ſometimes the common Diviſor thus diſcover'd is not the greateſt; which, when it happens, and how in ſuch a caſe to find the greateſt, I will ſhow by an Example.

EXAMPLE III.

There are Quantities given, order'd according to the Dimenſions of the Letter *a*.

$$\begin{array}{c} adc-dbc \\ afc-fbc \\ add-bdd \\ adf-fbd \end{array}$$

And

$$\begin{array}{c} agc-bgc \\ agd-bgd \\ alc-blc \\ ald-bld \end{array}$$

The greateſt common Diviſor of theſe is ſought.

I ought to divide the firſt Quantity by the ſecond (74).

But this Diviſor may be reduced (75), becauſe both Terms are diviſible by $gc+gd+lc+ld$, and the Quotient is $a-b$. By which the Diviſion of the firſt Quantity proceeds without a Remainder; yet $a-b$ is not the greateſt common Diviſor.

78. *For the greateſt common Diviſor is never found, when in any Diviſion the Quantity by which*

we reduce the Diviſor, is alſo the Diviſor of the Dividend, or has a common Diviſor with the Dividend; as in this Caſe, *in which*

$$gc + gd + lc + ld$$

can be divided by $c + d$; by which alſo the firſt of the given Quantities may be divided.

79. That we may therefore attain to the greateſt Diviſor, *we ought to examine in each ſingle Diviſion, whether the Quantity by which we reduce the Diviſor has the ſame common Diviſor with the Dividend of that Diviſion: If ſo, we are then to divide this Dividend by that common Diviſor, to reduce it; and when the whole Operation is finiſh'd, we muſt multiply the common Meaſure found by this particular Diviſor.*

In the laſt Example, when I find the Diviſor may be reduced by dividing it by $gc+gd+lc+ld$, I inquire whether this Quantity has not a common Diviſor with the Dividend; that is, with the firſt of the propos'd Quantities: And I find that $c + d$ is ſuch a common Diviſor, by which I divide that firſt Quantity, which is then reduced to this

$$\begin{array}{c} ad - bd \\ af - bf \end{array}$$

I had before reduced the Diviſor to $a - b$, which becauſe it divides the other without a Remainder, is the greateſt common Meaſure of theſe Quantities; I multiply therefore $a - b$ by $c + d$, and $ac + ad - bc - bd$ will be the greateſt common Diviſor of the Quantities propos'd.

OF THE REDUCTION OF FRACTIONS TO A COMMON DENOMINATION.

80. *Different Fractions are reduced to the ſame Denominator by multiplying the Numerator and Deno-*

minator of each, by the Denominators of all the reſt; by which Operation the particular Fractions are not changed (67).

EXAMPLE.

Fractions to be reduced,

$$\frac{ab}{f},\quad \frac{cbd}{hl},\quad \frac{q}{r}.$$

The ſame Fractions reduced are,

$$\frac{abhlr}{fhlr}\quad \frac{cbdfr}{fhlr}\quad \frac{qfhl}{fhlr}$$

81. *If two or more Denominators have a common Diviſor, theſe before the Operation may be divided by it. If then the Operation being ended, the common Denominator, as alſo the Numerators of thoſe Fractions which were not divided are multiplied by that common Diviſor, the reduced Fractions will be more ſimple.*

EXAMPLE.

Fractions to be reduced,

$$\frac{l}{s},\quad \frac{abd}{lh},\quad \frac{f^3}{mh}:$$

The common Diviſor of two Denominators is h and

$$\frac{llmh}{slmh},\quad \frac{abdsm}{slmh},\quad \frac{f^3sl}{slmh},$$

are the Fractions reduced. The Example demonſtrates it.

OF ADDITION AND SUBTRACTION OF FRACTIONS.

82. *When the Fractions are reduced to a common Denominator, let the Numerators be added or ſubtracted,*

Ex-

EXAMPLES.

Given $\frac{ad}{b}$, $\frac{fs}{b}$, $\frac{ml}{t}$,

The Sum is $\frac{adht + fsbt + mlbb}{bbt}$.

The Difference of the two firſt, if ſubſtracted, is $\frac{adh - fsb}{bh}$.

Given $\frac{afr}{ad + bd}$ $\frac{l^3 - mb^2}{aa - bb}$;

The Sum is $\frac{aafr - afrb + dl^3 - dmb^2}{aad - bbd}$.

The Difference is $\frac{aafr - afrb - dl^3 + dmb^2}{aad - bbd}$.

OF MULTIPLICATION.

The Multiplication of Fractions is founded upon this Theorem.

83. *The Product of any two Quotients is found by dividing the Product of the Dividends by the Product of the Diviſors.*

Let *ab* be the Dividend, the Diviſor *a*, the Quotient is *b*. Again, let *cd* be the Dividend, the Diviſor *c*, and Quotient *d*; theſe Quantities may repreſent all poſſible Quotients, Diviſors, and Dividends. The Product of the Quotients is *bd*: the ſame Quantity is found, if *abcd* the Product of the Dividends is divided by *ac*, the Product of the Diviſors.

84. *Fractions* are Quotients, *and are multiplied by multiplying their Numerators and Denominators* (38. 83.)

EXAMPLE.

$\frac{af}{d}$ by $\frac{as}{m}$ produces $\frac{aafs}{dm}$.

$\frac{bm}{a+b}$ by $\frac{lm}{a-b}$ produces $\frac{blm}{aa-bb}$.

85. That the Fraction which expreſſes the Product may be the ſimpleſt poſſible, it is not ſufficient to reduce the Fractions to their ſimpleſt Terms before Multiplication. For if the Numerator of one has a common Diviſor with the Denominator of another, the Product may be reduc'd by the following Rule.

86. *The Fractions being reduc'd to their ſimpleſt Terms, let the Denominators be chang'd, and then reduc'd again.* By this change the Fractions are indeed alter'd, but the Product is ſtill the ſame.

EXAMPLE.

Let there be Fractions to be multiplied

$$\frac{da^4 - db^4}{fgaa + fgbb}, \quad \frac{fl^3}{aal + 2alb + lbb}.$$

The Fractions reduced will be

$$\frac{daa - dbb}{fg}, \quad \frac{fl^2}{aa + 2ab + bb}.$$

And changing their Denominators, they are

$$\frac{daa - dbb}{aa + 2ab + bb}, \quad \frac{fl^2}{fg}.$$

And again reduced they are

$$\frac{da - db}{a + b}, \quad \frac{l^2}{g}.$$

The Product is $\frac{dal^2 - dbl^2}{ag + bg}$.

87. *If a Fraction is multiplied by the Denominator, the Product is the Numerator* (35. 40. 41.)

OF DIVISION.

88. *The Diviſion of Fractions is the Diviſion of the Quotients of 2 Diviſions* (38. 39.) *which are expreſſed by retaining the Dividends and Diviſors.*

Let *abc* be any Dividend whatſoever, *a* the Diviſor, the Quotient is *bc*; which is to be divided by the Quotient of this other Diviſion *cd* by *d*, which is *c*. The Quotient *bc* divided by *c* gives *b* for a Quotient, which we may alſo diſcover by a ſingle Diviſion.

89. *We muſt multiply the Dividend of the firſt Diviſion by the Diviſor of the ſecond, and divide the Product* abcd *by* acd *the Product of the Dividend of the ſecond Diviſion drawn into the Diviſor of the firſt, and we ſhall have* b *for the Quotient of the other two Quotients. The Diviſions here made uſe of, repreſent all poſſible Diviſions.*

Let there be now Fractions $\frac{a}{b}$ and $\frac{d}{c}$ of which the firſt is to be divided by the ſecond; the Dividend of the firſt Diviſion is *a*, the Diviſor *b*, the Quotient is the Fraction $\frac{a}{b}$ (38. 39.) the Dividend of the ſecond Diviſion is *d*, the Diviſor *c*, and the Quotient $\frac{d}{c}$.

If now we divide *ac* by *db*, we ſhall have the Quotient of the Fractions $\frac{ac}{db}$ (89.)

90. *We may reduce this Diviſion of Fractions to their Multiplication, if we change the Numerator of the Diviſor into a Denominator.*

The Diviſor is $\frac{d}{c}$, which chang'd is $\frac{c}{d}$; and this Fraction, if multiplied by $\frac{a}{b}$, gives the Quotient already diſcover'd $\frac{ac}{db}$.

It is manifeſt that all which has been obſerv'd relating to the Multiplication of Fractions (84. 85) may be here applied after the Tranſpoſition of the Numerator.

OF THE EXTRACTION OF ROOTS.

The Square of the Fraction $\frac{a}{b}$ is $\frac{aa}{bb}$, the Cube $\frac{a^3}{b^3}$; whence it follows,

91. *That the Root of a Fraction will be found by ſeparately extracting the Root from the Numerator and Denominator:* wherefore hither may be referr'd all that has been explain'd above (58, *&* *ſeq.*) relating to the Extraction of Roots.

EXAMPLES OF SQUARE ROOTS.

$$\frac{aa + 2ab + bb}{bb}$$

Has the Root $\frac{a + b}{b}$.

$\frac{bb}{df}$ has the Root $\frac{b}{\sqrt{df}}$,

$\frac{fd + gh}{dm}$ hath the Root $\frac{\sqrt{fd + gh}}{\sqrt{dm}}$ or $\sqrt{\frac{fd + gh}{dm}}$.

CHAP.

CHAP. V.

Of Surd Quantities.

92. S*Urd Quantities may often be reduced to more ſimple ones,* which Reduction is founded on this Theorem :

93. *The Root of a Product is had by multiplying the Roots of the Quantities to be multiplied.*

It is evident, if two Quantities aa and bb are to be multiplied, that their Roots are a and b, and that the Product of the Roots ab is the Root of the Product a^2b^2.

94. Let there be now a ſurd $\sqrt{aabd}$ this may be reduced, for it is the Root of the Product of aa by bd, the Roots of which Quantities are a and $\sqrt{bd}$, and the Product of theſe $a\sqrt{bd}$ is equivalent to the Root propos'd (93.) In the ſame manner $\sqrt[3]{a^3b}$ is equivalent to $a\times\sqrt{b}$.

95. In this caſe, the Quantity whoſe Root is expreſſed by the radical Sign, (as here b) is ſaid to be UNDER THE SIGN, and the other as a WITHOUT THE SIGN.

96. *Roots of the ſame Power, which when they are reduc'd to the ſimpleſt poſſible, have the ſame Quantity under the Sign, are called* COMMUNICANTS, as $\sqrt{aab}$ and $\sqrt{ccb}$, which are reduced to theſe, $a\sqrt{b}$, $c\sqrt{b}$.

97. *The radical Sign may ſometimes vary without changing the Root it expreſſes,* for it is evident that $\sqrt{aa}$, $\sqrt[3]{a^3}$, $\sqrt[4]{a^4}$, $\sqrt[5]{a^5}$, &c. do all expreſs the ſame Root a.

98. *This Mutation may be made by raiſing the Quantity under the Sign to a Power, by whoſe Index the Index of the radical Sign is multiplied :* ſo $\sqrt[3]{a^3}$ is not changed, if whilſt the Index of the Sign is multiplied by 2, to make it $\sqrt[6]{}$, a^3 is rais'd up to the ſecond Power, and is transform'd into a^6, $\sqrt[6]{a^6}$ being equivalent to $\sqrt[3]{a^3}$.

99. Hence we deduce the Rule by which *various Roots are reduced to a common radical Sign.*

100. But firſt we muſt teach how to diſcover a *Number the leaſt of all that can be exactly divided by every Index of the radical Signs given* ; for this *is the Index of the Sign ſought*, if we want the ſimpleſt Reduction.

101. *The Product of all the Indexes is* indeed *exactly diviſible by each Index, but if any amongſt them have a common Diviſor*, ſuch a Product is not the leaſt that can be divided by each. In this Caſe *all thoſe* Indexes which have a common Diviſor, *are to be divided by it (one only excepted) and the Quotients are to be made uſe of in the Multiplication, inſtead of the Indexes they proceed from.*

A Quantity, the leaſt of all that is divided by theſe five, *ab*, *df*, *alm*, *ln*, *ag*, is thus diſcovered : Of the three Quantities *ab*, *alm*, *ag*, which have *a* for a common Diviſor ; let two, as *ab* and *alm*, be divided by *a*, and we ſhall have *b*, *df*, *lm*, *ln*, *ag*. Amongſt theſe, two *lm*, *ln*, have a common Diviſor *l* ; let one of them be divided by *l*, and the given Quantities are reduced to theſe, *b*, *df*, *m*, *ln*, *ag*, and that the Product of theſe *bdfmlnag* is diviſible by each of the given Quantities, is evident from the Operation it ſelf.

102. *When the common radical Sign is diſcover'd, every Root is reduced to it, by dividing the common radical Sign by the Index of the Root, and*

raiſing the Quantity under the Sign to a Power, whoſe Index is the Quotient of that Diviſion. This follows from what has been already explained (98).

EXAMPLE.

$$\sqrt{ab},\quad \sqrt[3]{ab^2},\quad \sqrt[4]{a^3d},$$

Are reduced to

$$\sqrt[12]{a^6b^6},\quad \sqrt[12]{a^4b^8},\quad \sqrt[12]{a^9d^3}.$$

In the following Operations, it is always ſuppoſed that the Roots are reduc'd to a common radical Sign. We ſhall now explain Multiplication and Diviſion; and after that, Addition and Subtraction.

OF MULTIPLICATION.

The Multiplication of Roots is founded upon this Theorem:

103. *If the Root of the Product of the Powers is extracted, it will be the Product of the Roots.*

aa and bb may repreſent any Squares; the Roots of them are a and b, and the Product of theſe is ab.

The ſame Product is alſo found by multiplying aa and bb together, and extracting the Root of the Product $aabb$, nor is the Demonſtration different in any other Power.

104. *In Multiplication keep the radical Sign of the Roots, and multiply the Quantities under the Signs.*

EXAMPLES.

$\sqrt{a}$ by $\sqrt{b}$ gives $\sqrt{ab}$

$\sqrt{a+b}$ by $\sqrt{a-b}$ gives $\sqrt{aa-bb}$.

105. *If the Roots are multiplied by rational Quantities*, that is, if there are Quantities given without the Sign, *theſe muſt be multiplied ſeparately.*

EXAMPLES.

$a\sqrt{b}$ by $d\sqrt{c}$ gives $ad\sqrt{bc}$

$a\sqrt{b}$ by $e\sqrt{b}$ gives $ae\sqrt{bb}$ or aeb.

106. *When there are compound Quantities to be manag'd, the Rules deliver'd above* (45, 46.) *will be of uſe.*

Of DIVISION.

107. As Diviſion reſolves what Multiplication had compounded, it is evident, *That in Diviſion of Roots, the Quantities under the Signs are to be divided, and the Sign ſtill to be kept.*

EXAMPLES.

$\sqrt{ab}$ by $\sqrt{a}$ gives $\sqrt{b}$ (35. 104.)

$ad\sqrt{bc}$ by $a\sqrt{b}$ gives $d\sqrt{c}$ (35. 105.)

108. *In compound Quantities, the Diviſion is performed by parts,* as we have above explain'd.

EXAMPLE.

$$\begin{array}{l|l} adb + a\sqrt{fghb} - db\sqrt{df} - f\sqrt{dbgh} & \dfrac{a\sqrt{b} - \sqrt{dbf}}{d\sqrt{b} + \sqrt{fgh}} \\ \hline -adb \qquad\qquad + db\sqrt{df} & \\ \hline \quad + a\sqrt{fghb} \qquad\qquad - f\sqrt{dbgh} & \\ \quad - a\sqrt{fghb} \qquad\qquad + f\sqrt{dbgh} & \\ \hline \qquad\quad 0 \qquad\qquad\qquad\quad 0 & \end{array}$$

OF THE EXTRACTION OF ROOTS.

109. *Extraction of Roots is perform'd either by extracting the Root of the Quantity under the Sign, or by changing the Index of the radical Sign, or by annexing a new radical Sign.*

The ſquare Root of $\sqrt[3]{aa}$ is $\sqrt[3]{a}$.

The ſquare Root of $\sqrt[3]{ab}$ is $\sqrt[6]{ab}$.

The ſquare Root of $a+\sqrt{bd}$ is $\sqrt{a+\sqrt{bd}}$.

Other things that might be obſerv'd relating to theſe Extractions, are of a higher nature.

OF ADDITION AND SUBTRACTION.

110. *In Addition and Subtraction of Roots what has been already ſaid relating to ſimple and compound Quantities* (14. 43. 15. 44.) *is to be obſerv'd.*

111. But it is to be remark'd, that the Sum may be otherwiſe expreſſed; for the Sum of the Roots $\sqrt{d}$ and $\sqrt{f}$ is indeed $\sqrt{d}+\sqrt{f}$, but the ſame Sum is alſo $\sqrt{d+f+2\sqrt{fd}}$.

The Difference of the ſame Roots is $\sqrt{d}-\sqrt{f}$, or $\sqrt{d+f-2\sqrt{fd}}$.

The Demonſtration of this will be eaſy, if we conſider that the Sum of two Quantities is found, if we extract the ſquare Root from the Square of the Sum; but the Square of the Quantity $\sqrt{d}+\sqrt{f}$ is $d+f+2\sqrt{df}$.

In the ſame manner, the Difference of the ſame Quantities is found by extracting the ſquare Root of the Square of their Difference.

Something of the ſame nature might be ſhown as to Cubic and other Roots, but it would not be of any great uſe.

112. The Sum of Communicant Roots may be expreſſed otherwiſe; for $a\sqrt{b} + d\sqrt{b}$ is $\overline{a + b}\sqrt{b}$.

113. But whether the Roots are Communicants or not, is diſcover'd by Multiplication. If the Queſtion is of ſquare Roots, the ſquare Root will be exactly extracted from the Product of the Quantities under the Signs.

Let the Quantities be $\sqrt{aab}$ and $\sqrt{ddb}$, the Product $aabddb$ has a ſquare Root abd.

114. In Cubic Roots let one of the Quantities be rais'd to the Square, and let this be multiplied by the other Quantity; from which Product the Cubic Root may be extracted, when the Roots are Communicant.

Let there be $\sqrt[3]{a^3b}$ and $\sqrt[3]{d^3b}$, I multiply the Square a^6bb by d^3b, and the Product $a^6d^3b^3$ has a Cubic Root $aadb$.

115. In Roots of the fourth Power let one Quantity be rais'd to the Cube, and let this Cube be multiplied by the other Quantity.

116. And generally if the Index of the Root is n, one of the Quantities muſt be rais'd to a Power, whoſe Index is $n-1$, as is ſufficiently manifeſt from what hath been already explain'd.

* * *

CHAP. VI.

Of Proportions and Progreſſions.

117. THE Sign of Equality is this $=$.

We have obſerv'd in the firſt Chapter, that all Compariſon of Quantities may be referred to Addition, Subtraction, Multiplication or Diviſion.

118. Addition and Subtraction are oppoſed to each other, as alſo Multiplication and Diviſion. *Wherefore all that relates to the Compariſon of Quantities may* commodiouſly *be referred to two Claſſes.*

For in the Compariſon of two Quantities, we either regard their Difference, that is, what is to be added to one to form the other, or we conſider how often one is contain'd in the other, that is, by what Number (which may be a Surd) we muſt multiply one to produce the other.

There are various things to be obſerv'd about theſe Compariſons, before we come to the general Rules by which Truth is inveſtigated, and Problems are ſolved.

119. *If four Quantities being given, the Difference of the two firſt is equal to the Difference of the two laſt, thoſe Quantities are ſaid to be in Arithmetic Proportion.*

120. This is mark'd with three Points $\because$ thus, as $5,7 \because 13,15$. or $a,b \because c,d$.

121. In every Proportion the firſt and laſt Quantities are called EXTREMES, the others are called MEANS.

122. In this *Arithmetic Proportion*, $a,b \because c,d$, from the Nature of the Proportion $a - b = c - d$ (119),

(119), if to both we add $b + d$, we do not ſtroy the Equality; and then $a + d = c + b$, t is, *the Sum of the Extremes is equal to the Sun the Means.*

123. *Arithmetic Proportion is ſaid to be contin: when the firſt Term differs as much from the ſeco: as that does from the third,* as $f, g \because g, h$; wh. Proportion is alſo thus expreſſed ÷ f, g, h.

124. *A Series of Quantities, in which the Dif rence of any two directly following each other is alw. the ſame, is call'd an Arithmetic Progreſſion*; 6, 9, 12, 15, 18; theſe are mark'd by the ſan Sign ÷ as ÷ a, b, c, d, e.

125. This Progreſſion, if the Difference (the neighbouring Terms is n, may be thus ex preſſed: ÷ $a, a + n, a + 2n, a + 3n, a + 4$ &c. And *we ſee how from the two firſt Terms give any other may be found*; as for example, the tent. is $a + 9n$.

In the forementioned Progreſſion we have theſ Proportions, $a, b \therefore f, g$, and $b, c \therefore e, f$; and alſ $c, d \therefore d, e$. From hence we deduce, that $a + g = b + f = c + e = 2d$ (122); therefore $a + b + c + d + e + f + g = 3\frac{1}{2} \times a + g$, whence we form this general Rule.

126. *The Sum of an Arithmetic Progreſſion is equivalent to the Product of the Sum of the firſt and laſt Terms, by half the number of Terms.*

127. The other Compariſon of Quantities, of which we have ſpoken (118), is determin'd by Diviſion; for the Quotient expreſſes how often one Quantity is contain'd in another, and this Relation is call'd the RATIO.

128. We diſcover the Ratio given betwixt a and b, if we divide a by b, and the Quotient is called the EXPONENT of the Ratio.

129. We divide the firſt Quantity by the ſecond, to determine the Ratio; $\frac{a}{b}$ is the Exponent of the Ratio betwixt a and b; and on the contrary, $\frac{b}{a}$ is the Exponent of the Ratio of b to a.

130. The firſt Term of the Ratio is called the ANTECEDENT, and the ſecond the CONSEQUENT.

131. A Ratio is ſaid to be greater, by how much the Exponent is greater; and *vice verſâ.*

132. Ratio's are ſaid to be equal, if their Exponents are equal; for in that caſe both Conſequents are contained in the ſame manner in their Antecedents.

133. *Four Quantities, of which the firſt has the ſame Ratio to the ſecond, that the third has to the fourth, are ſaid to be in Geometrical Proportion*; or ſimply to form a Proportion.

134. This Proportion is thus mark'd ::

$$a, b :: c, d$$

ſignifies that a has the ſame Ratio to b, that c has to d.

Therefore if a is divided by b, the Quotient will be the ſame that ariſes from dividing c by d (127); that is, $\frac{a}{b} = \frac{c}{d}$.

If we multiply each Quantity by bd, we have $ad = bc$; whence it follows, that in *every Proportion the Product of the Extremes is equal to the Product of the Means.*

135. The Inverſe of this Propoſition is alſo true, *That Quantities are proportional, if the Product of the Extremes is equal to the Product of the Means.*

If $fg = hl$, I ſay $f, h :: l, g$.

For

For if we divide each of the Quantities by g, we have $\frac{f}{h} = \frac{l}{g}$, which demonſtrates the Equality of the Ratio's of f to h, and l to g.

136. *Of one Proportion there may be various others form'd.*

Let	$a, b :: c, d.$
By Inverſion	$b, a :: d, c.$
Alternately	$a, c :: b, d.$
Compounded	$a + b, b :: c + d, d.$
Divided	$a - b, b :: c - d, d.$
By Converſion	$a, a \pm b :: c, c \pm d.$

Theſe all ariſe from the firſt Proportion, $a, b :: c, d.$

That all theſe Proportions ariſe from the firſt, is very clear (135); for the Product of the Extremes appears by a ſimple Multiplication to be equal to the Product of the Means, if we only remember that $ad = bc$ from the firſt Proportion (134.)

137. In the ſame manner, we may prove that any two Proportions whatſoever being given, as

$$a, b :: c, d$$
$$f, g :: h, l.$$

will by Multiplication form this other,

$$af, bg :: ch, dl.$$

138. If the ſame Ratio has place in both Proportions, it will be alſo

$$a + f, \ b + g :: c + h, \ d + l.$$

139. *The Ratio betwixt two Quantities is not chang'd by their being multiplied or divided by the ſame Number*, it follows from a Theorem before demonſtrated (66).

140. *If there are given various equal Ratio's, the Sum of the Antecedents is to the Sum of the Conſequents, as one Antecedent to its Conſequent.*

Let it be $a, b :: c, d :: e, f :: g, h :: i, l$

Alter. $a, c :: b, d$.

Com. $a + c, c :: b + d, d$.

Alt. $a + c, b + d :: c, d :: e, f$.

Alt. $a + c, e :: b + d, f$.

Com. $a + c + e; e :: b + d + f, f$.

Alt. $a + c + e, b + d + f :: e, f$.

And by continuing it in the ſame manner, it will be evident,

$a + c + e + g + i, b + d + f + h + l :: i, l$, or a to b.

141. *If there are two Orders of Quantities, and the firſt is to the ſecond in the firſt Order as the firſt to the ſecond in the ſecond Order, and as the ſecond to the third in the firſt Order, ſo the ſecond to the third in the ſecond Order, and ſo of the reſt; then the Sum of all the Quantities in the firſt Order, will be to the Sum of all the Quantities in the ſecond Order, as any Quantity in the firſt Order to its correſponding Quantity in the ſecond.*

Let a, b, c, d, be the firſt Order.

f, g, h, i, the ſecond.

$a, b :: f, g$;

$b, c :: g, h$;

$c, d :: h, i$;

Therefore alter

$a, f :: b, g :: c, h :: d, i$;

And

$a + b + c + d, f + g + h + i :: a, f$, (140.)

A Ratio is ſaid to be the Inverſe of another, when the Antecedent of one is to its Conſequent, as the Conſequent of the other to its Antecedent. And a mutable Quantity is ſaid to follow the inverſe Ratio of another, when that is increaſed as this is diminiſhed, and *vice verſâ.*

142. *If there are given various Quantities, others that are as inverſely as theſe, are diſcover'd by dividing the ſame Quantity by each;* $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, are reſpectively inverſely, as *a*, *b*, *c*: for $\frac{1}{a}$, $\frac{1}{b}$ :: *b*, *a*, (135;) as is evident by multiplying the Extremes and Means.

143. *Three Quantities are ſaid to be in continual Proportion, if the firſt is to the ſecond, as the ſecond to the third.*

144. The Mark of ſuch a Proportion is ∺.
∺ *a*, *b*, *c*, ſignifies *a*, *b* :: *b*, *c*; and
∺ *a*, *b*, *c*, *d*, *e*, *f*, *g*, denotes
a, *b* :: *b*, *c* :: *c*, *d* :: *d*, *e* :: *e*, *f*, :: *f*, *g*.

145. Which Quantities are ſaid to be in GEOMETRIC PROGRESSION.

146. Of which this is a Property, *that the Sum of all the Terms except the laſt, are to the Sum of all except the firſt, as the firſt Term to the ſecond*, (140.)

147. *A Ratio is ſaid to be compounded of other Ratio's, when it is produced by multiplying thoſe Ratio's together. Ex. gr.* Let *a* be double *b*, and *c* triple *d*; now if *f* is both double and triple *g*, that is, twice triple, or three times double, or ſextuple; this laſt Ratio will be compounded of the other two.

148. *The Exponent of a compound Ratio, is the Product of the Exponents of the Ratio's it is compounded of.*

149. Whence we deduce, *that any Ratio's*, as *a* to *b*, and *c* to *d*, *f* to *g*, *being given Quantities in a Ratio compounded of theſe, will be diſcover'd by multiplying the Antecedents and Conſequents together;*

ther; the Exponent of the Ratio acf to bdg, is $\frac{acf}{bdg}$, the Product of the three Exponents of the precedent Ratio's $\frac{d}{b}$, $\frac{e}{d}$, $\frac{f}{g}$.

150. *A Ratio compounded of two equal Ratio's, is called a duplicate Ratio.*

151. *A triplicate Ratio is that which is compounded of three equal Ratio's; and ſo of the reſt.*

CHAP. VII.

Of Problems.

PRoblems are Queſtions to be ſolv'd, (11.)

152. *The* SOLUTION *of a Problem is* the Anſwer to a Queſtion, or *the Determination of the Quantities ſought.*

153. The ſame Problem has oftentimes various Solutions, from whence proceeds the Diſtinction of Problems.

154. *Thoſe Problems are indetermin'd which have innumerable Solutions.*

Ex. gr. If there are ſought 2 Numbers, whoſe Product is 24; the Problem will be indetermin'd, becauſe the Number of Solutions cannot be determin'd. For if we divide 24 by any Number at pleaſure, the Quotient will be a Number that, with the Diviſor, will anſwer the Queſtion.

155. *Determin'd Problems are thoſe,* which *have a determin'd Number of Solutions.*

156. *Thoſe which have only one Solution, are ſaid to be of one Dimenſion.*

Such is this:

Required 2 Numbers, whoſe Sum is 22, and Difference 6. Anſ. 14 and 8; neither is there any other Solution.

157. *Problems are ſaid to be of two Dimenſions when they have two Solutions.* Ex. gr.

To find a Number whoſe Square plus 6 is equal to the Number ſought multiplied by 5. Anſ. 2 or 3; nor hath it any other Solution.

158. The following *is a Problem of three Dimenſions.*

There is a Number ſought, which being multiplied by 11; *if* to the Product there is added *its* Cube, the Sum ſhall be equal to ſix times the Square of the Number ſought plus ſix. There are *three Solutions* of this Problem, and either of theſe Numbers 1, 2, 3, will anſwer the Conditions of the Problem; but no other.

159. In the ſame manner higher *Dimenſions are determin'd from the Number of Solutions.*

I ſhall here treat only of thoſe of one and two Dimenſions.

160. *All Problems are ſolv'd by Equations.*

161. Two *equal Quantities join'd by the Sign of Equality, are call'd an Equation.* This is ſaid to be of as many Dimenſions as are the Dimenſions of the higheſt Power of the unknown Quantity in it.

162. By the Rules deliver'd in the following Chapter, determin'd Problems are reduced to one Equation, which hath only one unknown Quantity in it, and this is called a SINGLE EQUATION.

163. In indetermin'd Problems, this laſt Equation contains two, or more unknown Quantities, which (one only excepted) may be determin'd

min'd at pleaſure; ſo that only one unknown Quantity remains in the ſingle Equation.

164. In this ſingle Equation, the unknown Quantity muſt be ſo ſeparated from the known, as to ſtand by itſelf in one Member of the Equation, or on one ſide of the Sign of Equality, and all the known Quantities on the other. This in Equations of one Dimenſion, is call'd REDUCTION of Equations; and in other Caſes the SOLUTION of Equations.

165. *The ſingle Equation has as many Dimenſions as the Problem has Dimenſions or Solutions.* This Propoſition, and in what Caſes there is an Exception to it, we will demonſtrate in what follows.

166. Some Problems, in which the unknown Quantity has two Dimenſions, are refer'd, as to their Solution, to Problems of one Dimenſion, and others in which the unknown Quantity has four Dimenſions, are referr'd to Problems of two.

167. Let x be unknown, and let there be given a and b, $xx + bb = dx$, is an Equation of two Dimenſions, and two Values of x ſatisfy the Conditions of this Equation.

But if $xx = ab$, it is referred to a Problem of one Dimenſion, and the Value of x is diſcover'd by a ſimple Extraction of the Root; for $x = \sqrt{ab}$. But it is nevertheleſs evident, that the Problem reduced to this Equation $xx = ab$ has two Solutions; for we have not only $x = \sqrt{ab}$, but alſo $x = -\sqrt{ab}$.

168. But ſuch Problems are referred to the firſt Dimenſion, becauſe they are ſolv'd by the ſame Rules; and this Caſe always holds, when

the ſame Quantity affirmative and negative, anſwers the Conditions of the Problem.

I ſhall treat in the tenth Chapter of Equations of four Dimenſions, that are referred to two.

CHAP. VIII.

Of the Rules by which Problems are ſolv'd.

169. THE *Reduction of an Equation is the ſeparating of one Quantity from the reſt; ſo that it alone ſhall be found in one Member:* In which caſe, the Value of it expreſſed by other Quantities, is neceſſarily contain'd in the other Member.

It is to be obſerved, that the Equality is not deſtroy'd, if both the Members are multiplied or divided by the ſame Quantity, nor if the ſame Quantity is added or ſubtracted from both.

170. Whence it follows, *That any Quantity may be tranſpoſed from one Member to another, by changing its Sign.*

By taking away from both an affirmative Quantity, or adding a negative on both ſides.

OF THE REDUCTION OF EQUATIONS.

RULE I.

171. *If the Quantity to be ſeparated is a Fraction, the whole Equation muſt be multiplied by its Denominator.*

Which may alſo be applied to any Number of Fractions.

RULE II.

172. *All the Quantities in which the Quantity to be ſeparated is found, are to be collected in one Member, all the others being tranſpoſed into the oppoſite one.*

This Tranſpoſition, as we have obſerved, (170) is done by changing the Signs.

RULE III.

173. *Let both Members be divided by the Quantity (whether ſimple or compound) that multiplies the Quantity to be ſeparated.*

Theſe three Rules are generally ſufficient in Equations of one Dimenſion, of which we here treat: But ſometimes this fourth comes in uſe.

RULE IV.

174. *If the Quantity to be ſeparated is involved with the radical Sign, this is firſt to be taken away by ſeparating according to the precedent Rules the Quantity thus involved from the reſt, ſo that it may be alone in one Member; and then raiſing both Members to the Power whoſe Root that Quantity is affected with.*

RULE V.

175. *If not the Quantity itſelf, but ſome Power of it is ſeparated; then the Root of this Power is to be extracted from both Members.*

EXAMPLE.

$$a + \frac{\sqrt{axx + fxx + d^3}}{\sqrt{g}} = b$$

$$\left.\begin{array}{l}\dfrac{\sqrt{axx+fxx+d^3}}{\sqrt{g}}=b-a\\[2ex]\dfrac{axx+fxx+d^3}{g}=bb-2ba+aa\end{array}\right\}\text{Rule 4.}$$

$$axx+fxx+d^3=gbb-2gba+gaa.\quad\text{Rule 1.}$$

$$axx+fxx=gbb-2gba+gaa-d^3.\quad\text{Rule 2.}$$

$$xx=\frac{gbb-2gba+gaa-d^3}{a+f}\quad\text{Rule 3.}$$

$$x=\frac{\sqrt{gbb-2gba+gaa-d^3}}{\sqrt{a+f}}\quad\text{Rule 5.}$$

OF THE REDUCTION OF VARIOUS EQUATIONS TO ONE: *Or of the Extermination of Unknown Quantities.*

RULE VI.

176. *Two Equations are reduced to one, by taking away one Quantity; which is done by ſeparating that Quantity ſo in one Equation, by the precedent Rules, that it may be found alone in one Member, and then ſubſtituting its Value in the other Equation.*

EXAMPLE.

Let there be Equations

$$ax+dy=fg$$
$$afx+ddx=ffy.$$

The firſt is reduced to this,

$$y=\frac{fg-ax}{d}$$

And ſubſtituting, in the ſecond Equation, for y, its Value $\frac{fg-ax}{d}$, we have this Equation, in which y is not given, $afx+ddx=\frac{f^3g-ffax}{d}$; which is reduced to this, $dafx+d^3x=f^3g-ffax.$

RULE VII.

177. *A Quantity may* alſo *be taken away by comparing its two Values together.*

EXAMPLE.

Let there be the ſame Equations,

$$ax + dy = fg,$$
$$afx + ddx = ffy$$

ſeparating y by the Rules of Reduction, we have

$$y = \frac{fg - ax}{d},$$

$$y = \frac{afx + ddx}{ff};$$

therefore $\frac{afx + ddx}{ff} = \frac{fg - ax}{d}$, which Equation multiplied by dff, gives the ſame we found by Rule VI. $dafx + d^3x = f^3g - ffax$.

RULE VIII.

178. *When there are given many Equations, one Quantity may be taken away by* Rule VI. & VII. *and the Number of Equations will then be diminiſhed by Unity.*

By Rule VI. ſeeking the Value of the Quantity to be exterminated in one Equation, and ſubſtituting that Value inſtead of that Quantity in the other Equations.

By Rule VII. ſeeking the Value of the ſame Quantity in all the Equations in which it is found, and comparing thoſe Values together.

RULE IX.

179. *By diminiſhing continually the remaining Equations by this Method, we ſhall at laſt come to one only Equation.*

Ex-

EXAMPLE.

$$\begin{array}{ll}
x+y+z=a & x=a-y-z \\
x+y+u=b & \\
x+z+u=c & \\
y+z+u=d & \\
\hline
a-z+u=b & z=a+u-b \\
a-y+u=c & \\
y+z+u=d & \\
\hline
a-y+u=c & y=a+u-c \\
y+a+2u-b=d & \\
\hline
2a+3u-c-b=d &
\end{array}$$

We will here add the Application of the ſecond Method to the ſame Example.

$$\begin{array}{ll}
x+y+z=a & x=a-y-z \\
x+y+u=b & x=b-y-u \\
x+z+u=c & x=c-z-u \\
y+z+u=d & \text{— — — — —} \\
\hline
a-z=b-u & z=a-b+u \\
a-y=c-u & \text{— — — — — —} \\
y+z+u=d & z=d-y-u \\
\hline
a-b+u=d-y-u & y=d-2u-a+b \\
a-y=c-u & y=a-c+u \\
\hline
d-2u-a+b=a-c+u &
\end{array}$$

The Methods hitherto explain'd can have place only when the Quantity to be exterminated is of one Dimenſion: I therefore add the two following Rules.

RULE X.

180. *If the Quantity to be exterminated is of many Dimenſions, then from the Equations in which it is found,*

found, there muſt be others form'd, by comparing together the different Values of its higheſt Power.

Theſe Values are found by the aſſiſtance of the Rules deliver'd about Reduction.

RULE XI.

181. *But when the higheſt Power is not the ſame in two Equations, one of the Equations is to be multiplied by the Quantity to be exterminated, or ſome Power of it.*

EXAMPLE.

Let there be ſought two Equations, in which x is not found from the given Equations.

$$axx + byx = yyz$$
$$dxx + fzx = aaz$$
$$exx + ddy = ffz$$

$$xx = \frac{yyz - byx}{a}$$
$$xx = \frac{aaz - fzx}{d}$$
$$xx = \frac{ffz - ddy}{e}$$

From theſe three Values of the Square of x, we form two Equations;

$$\frac{yyz - byx}{a} = \frac{aaz - fzx}{d}$$
$$\frac{aaz - fzx}{d} = \frac{ffz - ddy}{e}$$

The third which may be form'd out of theſe, is not to be accounted a new Equation.

But we want three Equations, becauſe there ought to remain two after x is exterminated.

I therefore take one of theſe laſt Equations, and ſeek from it the Value of xx, and compare it with one of the three precedent Values.

The firſt of the two laſt Equations gives $x = \frac{dyyz - a^3z}{dby - afz}$; and multiplying this Equation by x, produces $xx = \frac{dyyzx - a^3zx}{dby - afz} = \frac{ffz - ddy}{e}$.

And the Fractions being taken away by Multiplication, we have the three following Equations; from which x may be taken away, by reducing them to two (according to the Methods already explain'd, 176, 177.)

$$dyyz - dbyx = a^3z - afzx$$
$$ea^3z - efzx = dffz - d^3y$$
$$edy^2zx - ea^3zx = dbf^2yz - d^2byy - af^2z^2 + afddzy.$$

Of the Solution of Problems.

Rule XII.

182. *A Problem muſt be conſider'd abſtractly.*

Every Queſtion that can be propoſed about particular Quantities, may be applied to other Quantities, as is ſufficiently evident; and therefore the Solution will alſo take place in thoſe others: for this reaſon, every thing ought to be remov'd out of the queſtion, that is not neceſſarily to be conſider'd in the Solution. *Ex. gr.* ſuppoſe this Queſtion propoſed:

A Perſon being ask'd how old he was, anſwer'd, if 16 is ſubſtracted from thrice my Age, the Remainder will as much exceed an hundred, as my Age is ſhort of it. Qu. *What was his Age?*

In this Problem I do not conſider whether the Queſtion was really ask'd or not; I do not examine whether it is Years, or other Quantities, that enter the Queſtion; but, according to the above-mention'd Rule, I reduce the Problem to this.

A Number is ſought, from the Triple of which, if 16 is ſubſtracted, the Remainder will as much exceed an hundred, as an hundred does the Number it ſelf ſought.

RULE XIII.

183. *Any thing that is known relating to the Queſtion, tho' not propos'd, is to be made uſe of in ſolving it.*

I will illuſtrate this Rule by an Example. *Hiero* King of *Syracuſe* asked *Archimedes* what quantity of Silver was mixed with the Gold in his Crown; the Crown might be weigh'd, or otherwiſe examin'd, provided it was kept intire.

Every one ſees, that without the aſſiſtance of ſomething elſe known, the Problem could not be ſolv'd; but he compared the Content of the Crown with the Contents of a quantity of Gold and a quantity of Silver of equal weight with the Crown. This he did by immerging theſe Bodies in Water, and meaſuring the Water flowing over the Veſſel, which was fill'd with Water before they were put in. He then founded his Calculation upon the Contents thus diſcover'd. See another Solution of the ſame Problem, *vid. Prob.* 39.

RULE XIV.

184. *The Quantities made uſe of in the Conditions of the Problem, ought to be expreſſed by Letters.*

I muſt here obſerve the great Uſefulneſs of what has been remark'd above concerning the Diſtinction of Letters, by which the known and unknown Quantities are denoted (13).

RULE XV.

185. *Equations muſt be form'd from all the Conditions of the Problem.*

The Conditions of the Problem are either Equalities betwixt certain Quantities, that is, Equations; or other Relations that Equations may be form'd from, (122. 134.)

In order that a Problem may be determin'd, it is neceſſary that all the unknown Quantities ſhould be determin'd; that is, there are as many Determinations required, as there are unknown Quantities. Each Condition contains one Determination, and alſo produces one Equation: Therefore the Problem is not determin'd, unleſs there are as many Conditions, and conſequently Equations, as there are unknown Quantities.

RULE XVI.

186. *By the foregoing Rules, the Equations are to be brought to a ſingle one, which is it ſelf to be reduced or ſolv'd.*

CHAP.

CHAP. IX.

Of the Uſe of theſe Rules in Problems of one Dimenſion, and of avoiding ſome Difficulties which oftimes occur in Solutions.

THE Rules by which Truth is diſcover'd in the Mathematics being already explain'd, it will be neceſſary to illuſtrate the Uſe of them by ſome Examples.

And as Difficulties often occur, I will point out the Cauſes of ſome of them, and ſhew how ſometimes they may be avoided.

PROBLEM I.

187. *Two Merchants lay out each an equal Sum; the firſt gains* 126 fl. *the ſecond loſes* 87 fl. *The Sum of the firſt is now double the Sum of the ſecond:* Qu. *What did each lay out?*

If the Problem is abſtractly conſider'd, according to Rule 12, (182) it may be reduced to this.

To find a Number, to which if 126 *be added, it ſhall be double the ſame Number, when* 87 *is taken from it.*

Let the Sum ſought be called x, then the Condition of the Problem is this; $x + 126$ is equal to twice $x - 87$. Therefore

$$x + 126 = 2x - 174. \quad 300 = x.$$

But if we deſire an univerſal Solution, let a be the Gain, and b the Loſs,

$$x + a = 2x - 2b \qquad a + 2b = x$$

Whence we ſee, that however the Loſs and Gain may be varied, the Sum of the Gain, and double the Loſs, is equivalent to the firſt Sum ſought.

PROBLEM II.

188. *A Workman having received five Weeks Pay, and 6 Florins over, continues to work two Weeks longer: he then finds he had but a quarter of what he received left, and being paid for the other two Weeks, he finds he had* 21 *Florins:* Query, *What was his Pay per Week?*

By Rule 12. (182.) the Problem is reduced to this; *To find a Number, which multiplied by* 5 *and* 6 *added to the Product, and this Sum divided by* 4; *if to the Quote there is added twice the Number ſought, the Aggregate ſhall be* 21.

Let the Number ſought be x; this multiplied by 5 makes $5x$, adding 6 to it, we have $5x + 6$. Let it be divided by 4, $\frac{5x + 6}{4}$, and add to it twice x, and the Sum is equivalent to 21: that is,

$$\frac{5x + 6}{4} + 2x = 21$$

$$5x + 6 + 8x = 84$$

$$13x = 78$$

$$x = 6$$

PROBLEM III.

189. *Five Ells of Cloth coſt* 17½ fl. Qu. *What will fourteen Ells coſt?*

Let the Price ſought be x, it is evident that 5 is to 14, as 17½ to the Number of Florins ſought. Wherefore

$$5,\ 14 :: 17\tfrac{1}{2},\ x$$

$$5x = 14 \times 17\tfrac{1}{2}\ (134)$$

$$x = \frac{14 \times 17\frac{1}{2}}{5} = 49$$

190. From this Solution we deduce this general Rule: *When the three firſt Terms of a Geometrick Proportion are given, the fourth is found by multiplying the ſecond and third together, and dividing the Product by the firſt.* Which Rule, by reaſon of its great uſe, is called the Golden Rule, as alſo the Rule of Three, or the Rule of Proportions.

PROBLEM IV.

191. *A Merchant borrow'd ſome Money, and promiſed* 5 per Cent. per Annum *Intereſt.* Qu. *What Sum ought he to pay at ſeven months end, for the Uſe of* 665.

It is clear, that the Intereſt muſt follow the Proportion of the Time; and alſo of the Sum, that is the Intereſt, is in a Ratio compounded of theſe. Therefore

A Number is ſought, that may be to 5 in a compound Ratio of 7 Months to 12 Months; that is, in the Ratio of 7 to 12, and in the Ratio of 665 to 100. Conſequently, if x is the Intereſt ſought,

$$12 \times 100,\ 7 \times 665 :: 5,\ x$$

$$x = \frac{5 \times 7 \times 665}{12 \times 100} = 19\tfrac{19}{48}$$

PROBLEM V.

192. *Three Merchants gain'd in Fellowſhip* 1560 *Florins. The firſt laid in* 2000 *Florins, and at a year's end drew it out again; the ſecond put in* 660 *Florins for ſeven months; and the third* 1255 *for five months: How is the Gain to be divided?*

It is manifeſt, that the Gain is to be divided in the Ratio's of the Money each brought, and the Time they continued it in: therefore the Problem is reduced to this;

To divide 1560 in three parts, which may be betwixt themſelves, as 2000 × 12, 600 × 7, 1255 × 5.

Let theſe parts be x, y, z; now we have two Orders of Quantities, of which thoſe that are in the firſt Order follow the Ratio of thoſe that form the ſecond Order. We have beſides the Sums of both the Orders; for $x+y+z=1560$, and $2000\times12+660\times7+1255\times5=34895$.

Therefore (141)

$$34895, 1560 :: \left\{\begin{array}{l} 2000\times12=24000, x \\ 660\times7=4620, y \\ 1255\times5=6275, z \end{array}\right.$$

Hence, by three Rules of Proportion the unknown Quantities will be determin'd.

PROBLEM VI.

193. *Given the firſt, ſecond and laſt Term of a Geometrical Proportion, to find the Sum.*

Let a be the firſt Term, b the ſecond, d the laſt, and x the Sum; we have

$$a, b :: x-d, x-a \quad (146.)$$

$$bx-bd=ax-aa$$

$$x=\frac{bd-aa}{b-a}$$

PROBLEM VII.

194. *The Sum of the Ages of* John, James, Peter *and* Paul, *is* 98. *The Age of* Paul *is found by taking* 11 *from the Sum of the Ages of* Peter *and* John, *the Age of* James *taken from the Age of* Paul, *and* 3 *added, makes the Age of* Peter: *laſtly, the Age*

of John *is leſs by three than half the Sum of the Ages of* Peter *and* Paul Qu. *What is the Age of each?*

Let u be the Age of *John*, x the Age of *James*, y of *Peter*, and z of *Paul*.

The Conditions of the Problem abſtractly conſider'd, and algebraically expreſſed, give theſe Equations.

$$\begin{array}{ll}
u+x+y+z=98 & \\
u+y-11=z & \\
z-x+3=y & x=z-y+3 \\
y+z=2u+6 & \\
\hline
u+2z=95 & \\
u+y-11=z & y=z-u+11 \\
y+z=2u+6 & \\
\hline
u+2z=95 & 2z=95-u \\
2z+5=3u & \\
\hline
100=4u & u=25
\end{array}$$

Therefore,

$$\begin{array}{l}
2z=95-u=70 \quad \& \; z=35 \\
y=z-u+11=21 \\
x=z-y+3=17
\end{array}$$

OBSERVATION I.

195. Oftimes the Equations of the Problem are reduced to a ſingle one more eaſily than by the methods of *n.* 178, 179. but as this depends upon the particular Relation of the Equation, which may be varied to infinity, theſe more ſimple and elegant Solutions of Problems, which are founded upon ſuch particular Relations, can never be brought to fixed Rules. But we ought to note, that by the Addition or Subtraction of two Equations, a more ſimple one is oftimes form'd; at which in many caſes we ſhall alſo arrive, if

we compare two Values of the ſame compound Quantity together. Oftimes by theſe Operations alone ſome unknown Quantities are exterminated, as in the two following Examples.

PROBLEM VIII.

196. *There are two Quantities ſought, whoſe Sum and Difference is given.*

Let the Quantities be x and y, let the Sum be a, and Difference b; the Equations are

$$x + y = a$$
$$x - y = b$$

Theſe Equations being added, we have

$$2x = a + b$$

Or the ſecond ſubſtracted from the firſt,

$$2y = a - b$$

PROBLEM IX.

197. *There is given the Periphery* 80, *and Area* 240, *of a right-angled Triangle, to find the Hypothenuſe.*

Let the Sides be x, y, z; the Hypothenuſe x; the Equations given are

$$x + y + z = 80$$
$$yz = 480$$

To which, becauſe it is a right-angled Triangle; this third is to be added:

$$yy + zz = xx$$

If the ſecond Equation is multiplied by 2, and added to the third, we have $yy + 2yz + zz = xx + 960$.

The firſt Equation gives $y + z = 80 - x$.

The Squares of the Members are equal:

$$yy + 2yz + zz = 6400 - 160x + xx$$

Therefore $xx + 960 = 6400 - 160x + xx.$

$$160x = 5440$$
$$x = 34$$

OBSERVATION II.

198. Oftimes long and difficult Operations may be avoided, by ſeeking not the Quantities themſelves that are deſired, but others from whence they may be found.

This has place in all thoſe Caſes in which two unknown Quantities are found in the ſame manner, in all the Equations where either of them is.

Ex. gr. Let there be ſought two Numbers, whoſe Sum is a, and their Product b; if theſe Numbers are called z and u, the Equations of the Problem are

$$z + u = a$$
$$zu = b$$

It is evident in theſe Equations, that z cannot rather deſign the greater Number than the leſſer, and muſt therefore neceſſarily have a double Value in the Solution. It is the ſame if their Product is given and the Sum of their Squares, and in innumerable other Problems; In all theſe Caſes, notwithſtanding the Problem has only one Solution, we cannot attain to it without ſolving a Problem of two Dimenſions.

199. But in all theſe Caſes, when Problems arrive to two Dimenſions from this Cauſe only, as they do for the moſt part when Equations are not very much compounded; the Problems may be reduced to one Dimenſion, by ſeeking the Sum and Difference of thoſe unknown Quantities which have the ſame relation: Theſe being found, the Quantities themſelves are given, (196); but we always have the Caſe of *n.* 166, as to the unknown Quantity that repreſents the

Difference, becauſe the unknown Quantity may have a negative Value, but this is neceſſarily equal to the poſitive.

If, *ex. gr.* the Sum of two Quantities is called $2x$, and their Difference $2y$, the greater will be $x+y$, and the leſſer $x-y$, if y is a poſitive Quantity; but contrary, if it is negative. Put $x=7$, and $y=3$, then $x+y=10$, and $x-y=4$; but if $y=-3$, then it will be $x+y=4$, and $x-y=10$. Therefore in the Solution of the Problem y has two Values, 3 and -3; but it is but the ſame Quantity affirmative and negative, and therefore belongs to *n.* 166.

In the ſame manner x will ſometimes have two Values, which will be expreſſed by the ſame Quantity affirmative and negative, if the ſame Quantities being negative, ſatisfy the Conditions of the Problem, as in this:

Given the Product of two Numbers, and the Sum of their Squares, to find thoſe Numbers.

In this Problem, if the Numbers are called u and z, as u cannot expreſs one of the Numbers rather than another, u muſt have two different Values in the Solution; and theſe two Values ſuppos'd negative, will alſo anſwer the Queſtion. Wherefore after the Solution of a Problem of two Dimenſions, we have ſtill the Caſe of *n.* 166. remaining. But we may bring it to this by the Solution of a Problem of one Dimenſion, if one of the Quantities is called $x+y$, and the other $x-y$.

PROBLEM X.

200. *There are ſought two Quantities, whoſe Sum and Product is given.*

Let the Sum given be $2a$, and the Product b; let the Difference be $2y$.

The Quantities ſought are $a + y$ and $a - y$, (199) whoſe Product is $aa - yy$.

$$b = aa - yy$$
$$yy = aa - b$$
$$y = \sqrt{aa - b}$$

Problem XI.

201. *There are two Quantities ſought, their Product and Sum of their Squares being given.*

Let the Product be $2a$, and Sum of their Squares $4b$.

Let the Quantities be called $x + y$ and $x - y$, the Product then is $xx - yy$, and their Squares $xx + 2xy + yy$, and $xx - 2xy + yy$. Therefore

$$xx - yy = 2a$$
$$xx + yy = 2b$$

The Equations being firſt added, and then the firſt ſubſtracted from the ſecond, we have

$$2xx = 2a + 2b$$
$$2yy = 2b - 2a$$
$$x = \sqrt{a + b}$$
$$y = \sqrt{b - a}$$

Problem XII.

202. *There are three Numbers ſought in continual Proportion, their Sum and Sum of their Squares being given.*

The Sum of the Quantities is a, the Sum of their Squares b.

Let the Quantities be $\div s, u, z$.

The Conditions are $s + u + z = a$

$ss + uu + zz = b$

$sz = uu$, becauſe of their continual Proportion.

In this caſe s and z are found in the ſame manner in all the Equations; therefore that we may depreſs the Problem to one Dimenſion, let $x+y=s$, and $x-y=z$ (199.) then the Proportion ſought is $∺\ x+y,\ u,\ x-y$; and the Equations are

$$1.\quad 2x+u=a$$
$$2.\quad 2xx+2yy+uu=b$$
$$3.\quad xx-yy=uu$$

If in the ſecond Equation inſtead of uu we write its Value, we have

$$3xx+yy=b$$

And the third Equation being added,

$$4xx=b+uu$$

in 1^{st} $\quad 2x=a-u$

Therefore $4xx=aa-2au+uu=b+uu$

$$aa-2au=b$$

$$u=\frac{aa-b}{2a}=\frac{1}{2}a-\frac{b}{2a}$$

Therefore in 1^{st} $2x=a-u=a-\frac{aa+b}{2a}$.

$$x=\frac{aa+b}{4a}=\frac{1}{4}a+\frac{b}{4a}$$

In the 3^{d}

$$yy=xx-uu=\frac{a^4+2aab+bb}{16aa}-\frac{a^4+2aab-bb}{4aa}$$

$$yy=\frac{10}{16}b-\frac{3}{16}aa-\frac{3bb}{16aa}$$

$$y=\frac{1}{4}\sqrt{10b-3aa-\frac{3bb}{aa}}.$$

OBSERVATION III.

203. There often occurs a great Difficulty in the Solution of Problems with reſpect to the forming Equations from the given Conditions.

We have indeed remark'd above (185) that every Condition of a Problem gives an Equation; but oftimes the Equalities or Relations which ought to form thoſe Equations, are not directly expreſſed, although they are contain'd in the Conditions.

In this caſe we muſt examine whether there are not ſome Quantities conſider'd in the Problem, either known or unknown, which are not expreſſed in it: if there are ſuch, they muſt be deſign'd by Letters, and by the help of theſe Equations muſt be form'd, expreſſing the Relation of the unknown Quantities either amongſt themſelves, or with thoſe that are known.

204. If ſuch are not given, or do not ſuffice, we muſt examine whether the known or unknown Quantities mentioned in the Problem, have not Parts, the Relation of which either amongſt themſelves, or with other Quantities, may be diſcover'd from the Conditions of the Problem. If there are ſuch Parts, whether known or unknown, they are to be expreſſed by Letters; and from their Relations, Equations muſt be form'd.

PROBLEM XIII.

205. *A cubic Foot of Sea-ſalt Water weighs* a; *a cubic Foot of Rain Water* c; *there is ſome other ſalt Water, a cubic Foot of which weighs* b, *an intermediate betwixt* a *and* c. *Qu. What Weight of Rain Water muſt be added to a cubic Foot of Sea Water, that the Mixture may be equally ſalt with that mentioned?*

Let the Weight ſought be x; it muſt be remember'd, that the Bulk of Sea Water made uſe of in the Mixture is a cubic Foot, becauſe the Bulk is to be taken notice of, when the

Weights are determin'd. I alſo perceive, that the Bulks of the ſame Water are to one another as their Weights.

That I may therefore more eaſily arrive at Equations, I call z the Bulk of Rain Water, whoſe Weight is x; I alſo call a cubic Foot p. I have now relating to the Rain Water,

$$p, z :: c, x$$

And becauſe a cubic Foot of the Waters mixt weighs b, there is the ſame Ratio betwixt the Bulk p, and the whole Bulk $p + z$ of the Mixture, that there is betwixt the Weight b and the whole Weight $a + x$ of the ſame Mixture.

$$p, p + z :: b, a + x$$

Whence there are two Equations form'd, by which the Problem may be ſolv'd.

Theſe two Proportions may alſo be reduced to one, and the Problem to the Rule of Proportion, if we remark that from the laſt Proportion this may be deduced, (136.)

$$p, z :: b, a - b + x$$

Which compared with the firſt, gives

$$c, x :: b, a - b + x$$

By Alternation, Inverſion, and Diviſion, we have (136.)

$$b - c, c :: a - b, x$$

Problem XIV.

206. *There are three Meadows equally good* a, b, c, *in which the Graſs grows uniformly*; d *oxen in the time* e *eat up the Graſs in* a; f *oxen in the time* g *eat up the graſs in* b. *Qu. How many Oxen will eat up the Graſs in* c, *in the time* h?

Let the Number of Oxen ſought be x.

Let

The Conditions of the Problem do not expreſs the Relations, from whence Equations can be form'd. But becauſe regard is to be had to the Graſs which is in the Meadows, when the Oxen enter, and to the Graſs that grows afterwards, I diſtinguiſh the Oxen in each Meadow into two Parts.

I put $d=y+z$; the Oxen that feed on the firſt Graſs are y, and thoſe that feed on the Graſs that grows afterwards z. In the ſame manner I put $f=t+u$; and $x=s+r$.

By how much the Meadow is greater, ſo much the Number of Oxen (*cæteris paribus*) is greater; but by how much greater the time, by ſo much a leſſer Number of Oxen are required to eat up the ſame Graſs. Therefore the Conſideration of the Increaſe of the Graſs being ſet aſide, the Oxen are to one another directly as the Meadows, and inverſely as the Times: that is, y, t, s are amongſt themſelves directly as a, b, c, and inverſely as e, g, h.

The Oxen that feed on the Graſs that grows after they enter, are to one another as the Meadows, for the Time is not to be conſider'd; that is, z, u, r are to one another as a, b, c.

Theſe things laid down, the forming the Equations and the Solution is very eaſy.

$$d=y+z,\ \frac{a}{e},\ a$$

$$f=t+u,\ \frac{b}{g},\ b$$

$$x=s+r,\ \frac{c}{h},\ c$$

$$\frac{a}{e},\ \frac{b}{g}::y,\ t=\frac{eby}{ag}$$

$$\frac{a}{e},\ \frac{c}{h} :: y,\ s = \frac{ecy}{ab}$$

$$a,\ b :: z,\ u = \frac{bz}{a}$$

$$a,\ c :: z,\ r = \frac{cz}{a}$$

There are now given ſeven Equations, with ſeven unknown Quantities; and ſubſtituting in the ſecond and third for *t*, *s*, *u*, *r*, their Values which are found in the laſt four, theſe three Equations remain.

$$d = y + z \qquad y = d - z$$

$$agf = eby + gbz$$

$$ahx = ecy + hcz$$

$$agf = ebd - ebz + gbz \qquad z = \frac{agf - ebd}{gb - eb}$$

$$ahx = ecd - ecz + hcz \qquad z = \frac{ahx - ecd}{hc - ec}$$

$$\frac{agf - ebd}{gb - eb} = \frac{abx - ecd}{bc - ec} \quad x = \frac{acgfb - acgfe - bcedb + gbecd}{abbg - abbe}$$

OBSERVATION IV.

207. Sometimes it happens that the Conditions that determine the Problem, do indeed contain Equations; but in ſome Equations there is no unknown Quantity to be found. In ſuch a caſe thoſe Equations are of no uſe as to the exterminating the unknown Quantities, and cannot therefore immediately forward the Solution.

On theſe Occaſions, in the firſt place there muſt be new Equations form'd from the given ones, to which the above-mentioned Operations (176, &c.) may be applied. New Equations may always be produc'd from thoſe given, if (as is ſuppos'd) they contain a Determination of the Problem.

PROBLEM XV.

208. *Of various Mixtures given, to compound a new determined Mixture.*

Let there be given three Mixtures of Gold, Silver and Copper; the Quantity of Gold in the firſt Mixture a, in the ſecond b, in the third c; the Quantity of Silver in the firſt d, in the ſecond e, and in the third f; the Quantity of Copper in the firſt g, in the ſecond h, in the third i. There is to be a compound Mixture in which the Quantity of Gold is l, the Silver m, and Copper n; it is to be determin'd how much of each of the former Mixtures is to be taken.

Let Gold be called A; Silver B; and Copper C; the firſt Mixture M; the ſecond N; and third O; the Quantity ſought of the firſt Mixture x; of the ſecond y; and of the third Mixture z.

And that the Equations may be the more ſimple, I put

$$\begin{aligned} a+d+g&=o\\ b+e+h&=p\\ c+f+i&=q \end{aligned}$$

The Conditions of the Problem are contain'd in the following Equations.

$$\begin{aligned} aA+dB+gC&=oM\\ bA+eB+hC&=pN\\ cA+fB+iC&=qO\\ lA+mB+nC&=xM+yN+zO \end{aligned}$$

Only the laſt Equation contains unknown Quantities, the others therefore cannot contri-

bute

bute to the Solution, unleſs new ones can be form'd from them.

But this may be done by conſidering that l A is the Quantity of Gold given in the whole Mixture, that is, in xM, in yN, and zO; now this Gold may be otherwiſe expreſſed, if inſtead of M, N, and O, in the ſecond Member of the Equation are placed their Equivalents deſign'd by A, B, C.

I ſubſtitute therefore in the fourth Equation inſtead of M, N, and O, their Values deduced from the three firſt expreſs'd by A, B, C, and compare the Gold, Silver, and Copper in the firſt Member with the ſame Metal in the ſecond, by which means I form three Equations for the three unknown Quantities.

The three firſt Equations give

$$\frac{aA+dB+gC}{o}=M$$

$$\frac{bA+eB+hC}{p}=N$$

$$\frac{cA+fB+iC}{q}=O$$

And the laſt is changed into this

$$lA+mB+nC=\frac{xaA+xdB+xgC}{o}+\frac{ybA+yeB+ybC}{p}+\frac{zcA+zfB+ziC}{q}$$

From whence theſe three are deduc'd

$$l=\frac{xa}{o}+\frac{yb}{p}+\frac{zc}{q},\ x=\frac{opql-yboq-zcop}{apq}$$

$$m=\frac{xd}{o}+\frac{ye}{p}+\frac{zf}{q},\ x=\frac{opqm-yeoq-zfop}{dpq}$$

$$n=\frac{xg}{o}+\frac{yh}{p}+\frac{zi}{q},\ x=\frac{opqn-yhoq-ziop}{gpq}$$

Theſe three Values of the unknown Quantity x give two Equations, which will be more ſimple, if each is multiplied by pq, and then divided by o

$$\frac{pql - ybq - zcp}{a} = \frac{pqm - yeq - zfp}{d}$$

$$\frac{pql - ybq - zcp}{a} = \frac{pqn - yhq - zip}{g}$$

$$y = \frac{dpql - apqm + zafp - zdcp}{dbq - aeq}$$

$$y = \frac{gpql - apqn + zaip - zdcp}{gbq - ahq}$$

If theſe Values are multiplied by q, and divided by p, we have

$$\frac{dql - aqm + zaf - zdc}{db - ae} = \frac{gql - aqn + zai - zgc,}{gb - ah}$$

$$z = \frac{ahmq - aenq + dbnq - dhlq + gelq - gbmq}{ahf - aei + dbi - dhc + gec - gbf}$$

If in this Value of z we write p for q, b for c, e for f, and h for i, we ſhall have the Value of y.

But if for q we put o; for c, a; for f, d; and for i, g; we diſcover x. For if we attentively examine the Value of z, we ſhall eaſily perceive that 'tis only the Quantities q, c, f, i that diſtinguiſh the Value of z from the Values of y x, a, d, g, b, and e, h, y, z ſupplying their places in theſe Values.

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* * *

CHAP. X.

Of the Nature and Solution of Equations of two Dimenſions.

PRoblems of two Dimenſions have two Solutions (157) and the unknown Quantity two Values; therefore in the ſingle Equation which contains the Solution of the Problem, the unknown Quantity cannot rather have one of the Values than the other, and muſt conſequently contain them both.

209. Now that we may determine the Nature of Equations in which the ſame Letter deſigns two different Quantities, we muſt inquire into the Formation of ſuch an Equation.

210. Which that it may be done, we muſt conſider that all the Quantities of an Equation may be tranſpos'd to one Part of the Sign of Equality. This Equation $xx+6=5x$, by the Tranſpoſition of $5x$ is changed into this $xx-5x+6=0$.

211. If in this caſe inſtead of x we put its Value, the Quantities of the Equation will mutually deſtroy each other, becauſe they are equal to 0.

We muſt therefore ſhew how an Equation may be form'd, all whoſe Terms, if placed on the ſame ſide of the Sign of Equality, mutually deſtroy each other by the Subſtitution of either of the two Quantities.

212. Let $x=a$, and $x=b$, that theſe two Values may be comprehended in one Equation, I put $x-a=0$, and $x-b=0$; hence a new Equation is form'd by Multiplication.

$$\begin{array}{ll}
 & x-a=0 \\
 & x-b=0 \\ \hline
 & -bx+ba \\
xx-ax & \\ \hline
xx & -bx+ba=0 \\
 & -ax
\end{array}$$

If for x in this Equation I put either a or b, the Terms will mutually deſtroy each other.

That this may be demonſtrated, it muſt be obſerv'd, that it is the ſame thing whether I put for x its Value in the Product after the Multiplication is ended, or in the Quantities to be multiplied before Multiplication: It is alſo manifeſt that the Product form'd by the Multiplication of the ſame Quantity affirmative and negative deſtroys itſelf, as any Quantity multiplied by $+a-a$, or $+b-b$ is deſtroy'd, it being multiplied by o.

If in both Quantities now $x-a$, and $x-b$, we put a or b for x, and afterwards multiply, $b-a$ by $b-b$, or $a-b$ by $a-a$, in both caſes the Quantities of the Product deſtroy each other, and conſequently they do the ſame in the Equation.

213. The Value of the unknown Quantity is called the Root of the Equation, and it is evident that an Equation of two Dimenſions has two Roots.

214. And as a Negative Quantity may be deſign'd by a Letter, the Root is ſometimes negative; there are therefore four different Caſes.

1. Both the Roots may be affirmative; 2. one affirmative and the other negative, the affirmative being greater than the negative; 3. the negative greater than the affirmative; 4. both negative.

That we may determine the Forms of the Equations in theſe four caſes, I ſuppoſe a to be greater than b, and $a+b=d$, and $a-b=c$. Then I have theſe four Formula's.

$$\begin{array}{l} x=a\\ x=b\\ \hline x-a=0\\ x-b=0\\ \hline xx-ax+ab=0\\ \quad -bx\\ \hline xx-dx+ab=0 \end{array}$$

$$\begin{array}{l} x=a\\ x=-b\\ \hline x-a=0\\ x+b=0\\ \hline xx-ax-ab=0\\ \quad +bx\\ \hline xx-cx-ab=0 \end{array}$$

$$\begin{array}{l} x=-a\\ x=b\\ \hline x+a=0\\ x-b=0\\ \hline xx+ax-ab=0\\ \quad -bx\\ \hline xx+cx-ab=0 \end{array}$$

$$\begin{array}{l} x=-a\\ x=-b\\ \hline x+a\\ x+b\\ \hline xx+ax+ab=0\\ \quad +bx\\ \hline xx+dx+ab=0 \end{array}$$

215. We ſee hence how from the Signs alone of any Equation we may determine whether both the Roots are affirmative or negative, or one affirmative and the other negative; and alſo how to diſtinguiſh the greater of theſe from the leſſer.

216. It is alſo evident in every Equation of two Dimenſions, in which the Square of the unknown Quantity is ſeparated from all the known, and which has a Quantity entirely known on one ſide of the Sign of Equality, and which is alſo order'd according to the Dimenſions of the unknown Quantity; I ſay it is evident, that in ſuch an Equation the unknown Quantity in the ſecond Term is multiplied by the Sum of the Roots, its Sign being changed, and that the laſt Term is the Product of the Roots, its Sign preſerv'd.

The known Quantity that multiplies the ſecond Term, is call'd the COEFFICIENT of the ſecond Term.

OF THE SOLUTION OF EQUATIONS OF TWO DIMENSIONS.

The Rule for extracting the Root of ſuch a given Equation is this.

217. *Let the known Quantity be put alone on one ſide of the Sign of Equality, and add on both ſides the Square of half the Coefficient of the ſecond Term, and then extract the ſquare Root from both Members.*

The whole Artifice of this Rule is, that there is form'd a Square in the Member where the unknown Quantity is, whoſe Root may be exactly extracted.

EXAMPLE.

$$xx - dx + ab = 0$$

$$xx - dx = -ab$$

$$xx - dx + \tfrac{1}{4}dd = \tfrac{1}{4}dd - ab$$

$$\left.\begin{matrix} x - \tfrac{1}{2}d \\ \tfrac{1}{2}d - x \end{matrix}\right\} = \sqrt{\tfrac{1}{4}dd - ab}$$

$$x = \left\{\begin{matrix} \tfrac{1}{2}d + \sqrt{\tfrac{1}{4}dd - ab} \\ \tfrac{1}{2}d - \sqrt{\tfrac{1}{4}dd - ab} \end{matrix}\right.$$

It is clear, that both theſe Roots are poſitive, for $\sqrt{\frac{1}{4}dd - ab}$ is always leſs than $\frac{1}{2}d$.

It is alſo evident that the Roots of the Equation are always impoſſible when ab exceeds $\frac{1}{4}dd$; for then $\frac{1}{4}dd - ab$ is a negative Quantity, which cannot have a ſquare Root (46.)

OF PROBLEMS THAT PRODUCE EQUATIONS OF TWO DIMENSIONS.

We have remark'd ſomething above relating to theſe Problems (165.)

219. It is beyond all doubt, that Problems which have two Solutions produce an Equation of two Dimenſions (209, 212.)

220. We alſo light on ſuch an Equation when the Problem has indeed but one only Solution that we ſeek. But there is a certain negative Quantity that we are not at all concern'd with, which alſo ſatisfies the Conditions of the Problem: for Algebra does not lead us more readily to a poſitive than to a negative Solution; and therefore we diſcover both. But in this caſe we neglect the negative Root.

221. If the Problem has only two negative Solutions, neither of which we ſeek, we come to an Equation of two Dimenſions, notwithſtanding there is no Quantity that anſwers our purpoſe.

222. We may alſo come to a like Equation in a Problem altogether impoſſible. We have already ſhown, that an impoſſible Quantity may be Algebraically expreſſed (218). If ſuch Quantities Algebraically expreſſed anſwer the Conditions of the Problem, we arrive at an Equation whoſe Roots are impoſſible.

223. There is a particular Caſe worthy to be remark'd: this is, when a Problem has only one poſitive Solution; but in ſolving it, we have an Equation that has two poſitive Roots.

This happens when the Problem, beſides its poſitive Solution, has alſo a negative one, but the Quantity that belongs to the negative Solution is not itſelf negative.

224. *Ex. gr.* This Problem has only a poſitive Solution: *There is ſought a continual Proportion, whoſe firſt Term is* 4, *and the Difference of the firſt and ſecond*, 3. The poſitive Solution is ∺ 4, 6, 9; but there is alſo given this other negative one, ∺ 4, —2, 1, which we do not ſeek, but to which nevertheleſs Algebra brings us.

225. If we call the ſecond Term y, and the third x; it is clear, that when y is ſought, theſe

two Values are found, 6 and -2, but the negative is rejected. But if we ſeek x, we have the Values 9 and 1; where the laſt is to be rejected, becauſe it belongs to a Proportion in which there is a negative Term, which therefore we did not ſeek.

226. *If the firſt Term of the continual Proportion is 4, the Sum of the ſecond and third Term is 3, and the third Term is ſought.* We diſcover the ſame poſitive Quantities 9 and 1, but here 9 is to be rejected, becauſe it belongs to a Proportion which has a negative Term; for the Proportions are $\div 4, 2, 1$, and $\div 4, -6, 9$.

227. In the Solution of both theſe Problems, if the third Term is called x, and is ſought, we have this Equation $xx - 10x + 9 = 0$; ſo that one Problem cannot be reſolv'd without reſolving the other at the ſame time: yet both Solutions, to ſpeak mathematically, belong to both Problems, at leaſt to the Problems algebraically expreſſed.

228. In the Solution of this Equation $xx - 10x + 9 = 0$, we may diſtinguiſh the Root to be retain'd from that to be rejected, if we conſider the Operations by which we arrived at the Equation. To ſolve this Equation, it muſt be chang'd into the following,

$$xx - 10x + 25 = 16 \quad (127)$$

And the ſquare Roots extracted, we have

$$x - 5 = 4$$
$$5 - x = 4$$

229. The firſt of theſe Equations is to be retain'd, if in the preceding Operations the Square was form'd of x as an affirmative Quantity, as is done when the Problem of *n.* 224. is ſolv'd.

230. But the ſecond is to be retain'd, if, as in the Solution of Problem *n.* 224. the Square was form'd of the Quantity from which x was ſubtracted: for in this laſt Caſe xx in the Equation is not the Square of $+x$, but of the Root $-x$; and the Root $+x$ belongs to a Caſe about which we are not ſollicitous, although conſidering the thing abſtractly, it belongs to the Solution of the Problem.

231. But when the Operations do not determine whether the Square of the unknown Quantity has an affirmative or negative Root, both the Roots of the Equation of two Dimenſions give the Solution of the Problem, which has then two poſitive Solutions.

232. I have ſaid above (*166*) that ſometimes a Problem of four Dimenſions may be reduced to two, becauſe it is reſolv'd as a Problem of two.

233. This Caſe always happens if the ſame two Quantities affirmative and negative anſwer the Problem. *Ex. gr.* if $+a$, $-a$, $+b$, $-b$, ſatisfy the Conditions propos'd, the Problem has four Solutions. The Equation it is ſolv'd by, ariſes to four Dimenſions, but is ſolv'd by the Rules deliver'd for Equations of two Dimenſions, by ſeeking not the Value of the unknown Quantity itſelf, but of its Square.

For it is manifeſt in this Example, that if the Quantity ſought is called x, $x = +a$, and $x = -a$, is expreſſed by this Equation $xx = aa$; and in the ſame manner $x = b$ and $x = -b$ by $xx = bb$.

Theſe Equations being multiplied,

$$\begin{array}{l} xx - aa = 0 \\ \underline{xx - bb = 0} \end{array}$$

The Equation $x^4 \begin{matrix} -aaxx \\ -bbxx \end{matrix} + aabb = 0$ will contain four Values, of x (212), which (the Equation being given) are diſcover'd by the Rules deliver'd above (217) by ſeeking in the firſt place the two Values of the Square xx, and then extracting the Roots of them.

234. But we ſometimes light on ſuch an Equation, although the Problem has but one affirmative and an equal negative Solution. Then the Square of the unknown Quantity has indeed two Values; but one of them is negative, and to be rejected (218.)

Algebra neceſſarily gives all the Algebraic Solutions; but as impoſſible Quantities may be algebraically expreſſed, if theſe anſwer the Conditions of the Problem, ſuch Solutions will be neceſſarily contain'd in the Equation expreſſing the Values of the unknown Quantity.

235. There is alſo a Caſe analogous to that above mentioned in *n*, 223. for in a Problem that has only one affirmative, and an equal negative Solution, the unknown Quantity has ſometimes four true Values, but two of theſe belong to impoſſible Caſes, although the Values themſelves are not impoſſible.

EXAMPLE.

236. Let there be ſought x, y, z, ſuppoſing theſe following Proportions:

$$\div\ x,\ y,\ z$$
$$\div\ z-y,\ 20,\ z+y$$
$$\div\ 34-2y,\ z-x,\ 34+2y$$

This Problem has only one affirmative Solution, that is, $x=9$, $y=15$, $z=25$; which Numbers, if negative, give alſo the abovemention'd Proportions; that is, $x=-9$, $y=-15$,

-15, $z=-25$, and the Problem has no other Solution. But if I ſeek z, I have this Equation

$$z^4-689zz+40000=0$$

Which gives theſe four Values of z.

$$z=25$$
$$z=-25$$
$$z=8$$
$$z=-8$$

The two laſt Values belong to impoſſible Solutions, in which

$$x=-42,\ y=\sqrt{-336},\ z=8$$

Or $x=42$, $y=-\sqrt{-336}$, $z=-8$

237. But this Value of $z=25$ is to be diſtinguiſhed from thoſe that muſt be rejected by the following Method.

In the Solution of the Equation (217)

$$z^4-689zz+40000=0$$

I diſcover

$$\left.\begin{matrix} zz-344\frac{1}{2} \\ 334\frac{1}{2}-zz \end{matrix}\right\}=280\frac{1}{2}$$

I here chuſe the Value in the firſt Equation, rejecting the other, according to the Rule laid down above, (229, 230) becauſe in the Operations by which I came at a ſingle Equation, I form'd a Square of $zz-20$, and z^4 was produced by multiplying $+zz$ by it ſelf, not $-zz$. I have now $zz=625$, whence I find $z=25$, and $z=-25$; which laſt I reject, becauſe it is negative.

238. Oftimes Problems that have but one or two Solutions, cannot be reſolv'd but by Equations of three or four Dimenſions, and ſometimes of more. But in theſe Caſes, ſtrictly ſpeaking, the Problems have as many Solutions as the Number of Dimenſions, tho' ſome of them are rejected.

CHAP. XI.

Problems of Two Dimenſions.

I Will here only add a few Problems. If thoſe things explain'd in the precedent Chapter are added to what has been deliver'd before about Problems in general, the Solution of Problems of two Dimenſions will not be difficult.

PROBLEM XVI.

239. A *and* B *together owe* 208 ; A *pays each day* 9, *and* B *pays the firſt day* 1, *the ſecond day* 2, *the third* 3, &c. Qu. *How many days will the whole Debt be paid in, and how much does each pay?*

Let A's Debt be x, B's y, and the Number of Days z.

$$x + y = 208$$
$$9z = x$$

y is equivalent to the Sum of an Arithmetic Progreſſion, whoſe firſt Term is 1, laſt z, and Number of Terms z; which Sum is equal to $\overline{1+z} \times \frac{1}{2}z$ (126), therefore $\frac{1}{2}z + \frac{1}{2}zz = y$.

Which three Equations, by adding the two laſt together, and then comparing it with the firſt, are reduced to this :

$$x + y = \tfrac{1}{2}zz + 9\tfrac{1}{2}z = 208$$
$$zz + 19z = 416$$
$$zz + 19z + 90\tfrac{1}{4} + 506\tfrac{1}{4}$$
$$z + 9\tfrac{1}{2} = 22\tfrac{1}{2}$$
$$z = 13$$
$$x = 117$$
$$y = 91$$

PROBLEM XVII.

240. *A Perſon buys a Horſe, which he ſells again for* a, *and gains as much* per Cent. *as the Horſe coſt him.* Qu. *What did the Horſe coſt him?*

Let the Price of the Horſe be x.

$$100,\ 100 + x :: x,\ a$$

PROBLEM XVIII.

241. *There are two Numbers ſought, whoſe Product is* 12, *and the Difference of their Squares* 7.

Let the Numbers be x and y.

$$xy = 12,\ y = \frac{12}{x},\ yy = \frac{144}{xx}$$

$$xx - yy = 7,\ xx - \frac{144}{xx} = 7$$

$$x^4 - 144 = 7xx$$

This Equation is ſolv'd as an Equation of two Dimenſions (217), if we ſeek the Value of the Square of xx.

$$x^4 - 7xx = 144$$

$$x^4 - 7xx + 12\tfrac{1}{4} = 156\tfrac{1}{4}$$

$$\left.\begin{matrix} xx - 3\frac{1}{2} \\ 3\frac{1}{2} - xx \end{matrix}\right\} = 12\tfrac{1}{2}$$

$$xx = \begin{cases} +16 \\ -9 \end{cases}$$

$$x = 4,\ y = 3$$

I neglect the ſecond Value $xx = -9$, becauſe it is impoſſible (218).

I neceſſarily diſcover the Value of the Square xx in the Solution, becauſe not only $x = 4$, $y = 3$, but alſo $x = -4$, $y = -3$, ſolve the Problem.

I alſo diſcover two other Values of the Square xx, becauſe the Problem has beſides two impoſſible Solutions, which may be algebraically ex-

expreſſed: Theſe are $x = \sqrt{-9}$, $y = -\sqrt{-16}$, and $x = -\sqrt{-9}$, $y = -\sqrt{-16}$.

PROBLEM XIX.

242. *There is a Number ſought, to which if there is added the ſquare Root of itſelf multiplied by* 10, *the Sum ſhall be* 20.

Let the Number be x.

$$x + \sqrt{10x} = 20$$
$$\sqrt{10x} = 20 - x$$
$$10x = 400 - 40x + xx$$
$$xx - 50x = -400$$
$$xx - 50x + 625 = 225$$
$$\left.\begin{array}{r} x - 25 \\ 25 - x \end{array}\right\} = 15$$
$$xx = 40$$
$$x = 10$$

It is evident, that it is not the firſt Value that is ſought, becauſe it exceeds 20. I therefore take the ſecond.

But to ſpeak mathematically, 40 anſwers the Queſtion; for this multiplied by 10, produces 400, the ſquare Root of which -20, added to 40, the Sum $40 - 20$, is equal to 20.

It muſt be obſerv'd, that $25 - x = 15$, not $x - 25 = 15$, gives the Root that we ſeek, becauſe the Square was form'd of $20 - x$, (230.).

OBSERVATION V.

243. We have remarked above (198) that much labour may ſometimes be avoided in a Solution, by ſeeking not the Quantities themſelves that we want, but others from whence they may be found; and have explain'd the Rule which in many ſuch Caſes we may make uſe of (199). To this we may now add, that the Operations themſelves ſometimes point out the Quantities

that may more eaſily be diſcover'd than thoſe propoſed, and from whence thoſe ſought may be deduced: which will be more apparent from an Example.

PROBLEM XX.

244. *Given the Sum, and the Sum of the Squares of four Numbers in Geometric Progreſſion, to find thoſe Numbers.*

Let the Sum of the Numbers be a, and the Sum of their Squares b.

Let the Numbers be called $\div u, x, y, z$.

The Equations are

$$u + x + y + z = a$$

$$uu + xx + yy + zz = b$$

$$xx = uy \qquad u = \frac{xx}{y}$$

$$yy = zx \qquad z = \frac{yy}{x}$$

The four Equations are reduced to theſe two,

$$\frac{xx}{y} + x + y + \frac{yy}{x} = a$$

$$\frac{x^4}{yy} + xx + yy + \frac{y^4}{x^2} = b$$

And the Fractions being taken away, we have

$$x^3 + x^2y + yyx + y^3 = axy$$

$$x^6 + x^4y^2 + y^4x^2 + y^6 = bxxyy$$

In which x and y are affected in the ſame manner; wherefore I rather ſeek the Sum and Difference of theſe Quantities: but as I perceive by beginning the Computation, that it will be very operoſe; before I proceed any farther, I return to the Equations in which x and y are affected in the ſame manner, and conſider if it is poſſible to know from them what Quantities may be more eaſily diſcover'd; which will, when known, give x and y.

The forementioned Equations may be thus expreſſed.

$$\overline{xx+yy}\times\overline{x+y}=axy$$
$$\overline{x^4+y^4}\times\overline{x^2+y^2}=bx^2y^2$$

Which are changed into theſe;

$$xx+yy=\frac{axy}{x+y}.$$
$$x^4+y^4=\frac{bxxyy}{xx+yy}$$

The laſt, by ſubſtituting for $xx+yy$, its Value is chang'd into this;

$$x^4+y^4=\frac{bxy\times\overline{x+y}}{a}$$

That I may have a ſimpler Equation, I form the Squares of each Member of the firſt;

$$x^4+y^4+2yyxx=\frac{aaxxyy}{\overline{x+y}^2}.$$

Then I have

$$x^4+y^4=\frac{\overline{axy}^2}{\overline{x+y}^2}-\overline{2xy}^2=\frac{bxy\times\overline{x+y}}{a}$$

Which Equation I ſee would be more ſimple, if I put one unknown Quantity for xy, and another for $x+y$; that is, if I ſeek the Sum and Product of the unknown Quantities; which being given, the unknown Quantities themſelves are eaſily found (200.)

I muſt now ſee if I can find another Equation which alſo becomes more ſimple by the ſame Subſtitution; for two Equations are required to determine the two unknown Quantities.

There is given this Equation,

$$xx + yy = \frac{axy}{x+y}$$

$2xy$ added on both ſides, changes it to this,

$$\overline{x+y}^{2} = \frac{axy}{x+y} + 2xy.$$

Let us now put $xy = s$, and $x + y = t$, and Subſtitution being made, there are given theſe two Equations.

$$tt = \frac{as}{t} + 2s.$$

$$\frac{aass}{tt} - 2ss = \frac{bst}{a}$$

Which are changed into theſe,

$$t^{3} = as + 2st$$

$$a^{3}s - 2atts = bt^{3}, \quad t^{3} = \frac{a^{3}s - 2atts}{b}$$

Whence $as + 2st = \dfrac{a^{3}s - 2atts}{b}$

$$ba + 2bt = a^{3} - 2att$$

$$tt + \frac{b}{a}t = \tfrac{1}{2}aa - \tfrac{1}{2}b.$$

Which Equation being ſolv'd, gives t, and thence s; for $s = \dfrac{t^{3}}{a + 2t}$.

x and y are alſo diſcover'd,

$$x = \tfrac{1}{2}t \pm \sqrt{\tfrac{1}{4}tt - s}$$

$$y = \tfrac{1}{2}t \mp \sqrt{\tfrac{1}{4}tt - s}.$$

CHAP. XII.

Of Indetermined Problems.

WE have already remark'd that indetermin'd Problems may be ſolv'd like determin'd ones (*163*.)

But there are ſome indetermin'd Problems in which notwithſtanding there are more unknown Quantities than Equations, yet none of them can be aſſum'd at Pleaſure.

Such are all Problems in which Quantities are to be equal to Squares or Cubes.

In theſe, other Quantities are to be aſſum'd at Pleaſure, and from thoſe the indetermin'd ones of the Problem are to be determin'd.

When a Quantity whoſe Conditions are determin'd, ought to be equal to a Square or Cube, the whole Artifice is that the Quantity which is to be equal to a Square may be algebraically expreſſed, and put equal to a Square in ſuch a manner that the Equation with Reſpect to the unknown Quantity may be reduc'd to one Dimenſion.

As long as we are only concern'd with Problems of one and two Dimenſions, the following Rules are ſufficient in thoſe Caſes, where the Problems may be univerſally ſolv'd: When we cannot come at the Solution by theſe Rules, the Problem is only poſſible in ſome particular Caſe depending on the Property of the Numbers expreſſed in the Problem. Of theſe I ſhall not treat, becauſe I do not conſider the ſingular Properties of particular Numbers.

RULE I.

245. *When the Quantity to be determin'd is of one Dimenſion, the Root of the Square may be aſſum'd at Pleaſure.*

EXAMPLES.

ax ought to be equal to a Square, and x is the Quantity to be determin'd, taking v any Number at Pleaſure and putting $ax = vv$, x may be determin'd.

In the ſame manner x may be determin'd in theſe Equations.

$$b + ax = vv$$
$$b + cx + ax = vv$$

RULE II.

246. *When the Quantity to be determin'd has two Dimenſions, but the Square is affirmative and only multiplied by Unity, the Root of the Square is the Difference betwixt the Number taken at Pleaſure, and the Quantity to be determin'd.*

EXAMPLES.

Let $a + xx$ be a Quantity which ought to be equal to a Square, and x the Quantity to be determin'd, v a Number taken at Pleaſure, the Root of the Square is $v - x$. We have hence this Equation.

$$a + xx = vv - 2vx + xx$$
$$a = vv - 2vx.$$

In which x may be determin'd.

In the ſame manner x may be determin'd, if we have

$$a - bx + xx = vv - 2xv + xx$$
$$a - bx = vv - 2xv.$$

RULE III.

247. *When the Square of the Quantity to be determin'd is ſubſtracted in the Quantity propos'd, or when it is multiplied by another Quantity, the Problem cannot be ſolv'd univerſally. But only in thoſe caſes, when the Quantity that ought to be equal to the Square, conſiſts of a Square and other Quantities, in which the Quantity to be determin'd is found; then the Root of the Square ſought, is the Difference of the Product of the Quantity to be determin'd by a Number taken at Pleaſure, and of the Root of the Square that is given in the Quantity propos'd.*

EXAMPLES.

$aa - xx$ ought to be equal to a Square, x is the Quantity to be determin'd; v is taken at pleaſure, and the Root of the Square ſought is $vx - a$.

$$aa - xx = vvxx - 2vxa + aa$$

$$-x = vvx - 2va.$$

In which Equation x may be determin'd.

In the ſame manner in this Equation

$$aa + dx + exx = vvxx - 2avx + aa$$

$$d + ex = vvx - 2av.$$

248. It is to be obſerv'd, that o may be accounted a Square; the Rule therefore may be applied to this Quantity $dx - exx$, and the Root of the Square ſought ſhall be vx.

249. It is to be further obſerv'd, that thoſe Quantities may be referred to this Rule, where the Square of the Quantity to be determin'd is ſubſtracted from various Squares join'd together; as in this Quantity $aa + bb - xx$, where x is the Quantity to be determin'd.

Put $x = z - b$, therefore $xx = zz - 2zb + bb$, and the Quantity propoſed is changed into this, $aa + 2zb - zz$; which belongs to the third Rule: and the Value of z being diſcover'd, if b is ſubſtracted from it, we ſhall have x.

250. The above-mentioned Rule may alſo be applied to Quantities that contain Fractions. Let there be ſuch a Quantity $aa + \frac{bx - dxx}{c}$ let this be reduced to $\frac{aacc + bcx - dcx^2}{cc}$ (70.67) which may be made equal to a Square, if the ſame Numerator is a Square.

251. When various Quantities ought to be equal to a Square, let one be taken firſt, and made equal to a Square; and then making uſe of the Value of this Quantity ſo determin'd, let it be exterminated out of the other Quantities: And here thoſe Rules will be of uſe, that have been already laid down, relating to other Problems, as will be manifeſt in the Solutions of the following Chapter.

* * *

CHAP.

* * *

CHAP. XIII.

Indetermin'd Problems.

252. *TO find two Squares whoſe Difference is given.*

Let the given Difference be a, and the Squares ſought xx and yy.

$$a = xx - yy$$
$$a + yy = xx$$

That is, $a + yy$ ought to be equal to a Square; taking v at pleaſure, we have, by Rule 2. (246.)

$$x = v - y$$
$$a + yy = yy - 2vy + vv$$
$$2vy = vv - a$$
$$y = \frac{vv - a}{2v}$$
$$x = \frac{vv + a}{2v}$$

PROBLEM XXII.

253. *To divide a given Square into two other Squares.*

Let the given Square be aa, the Squares ſought xx, yy.

$$aa = xx + yy$$
$$aa - yy = xx$$

Therefore $aa - yy$ is to be equal to a Square: Now taking v at pleaſure, we have, by Rule 3. (247.)

$$x = vy = a$$

$$aa - yy = vvyy - 2avy + aa$$

$$-y = vvy - 2av$$

$$y = \frac{2av}{vv + 1}$$

$$x = \frac{avv - a}{vv + 1}$$

PROBLEM XXIII.

254. *To divide the Sum of two given Squares into two other Squares.*

Let the given Squares be aa, bb, and the Squares ſought xx, yy.

$$aa + bb = xx + yy$$

$$aa + bb - yy = xx$$

Therefore $aa + bb - yy$ ought to be equal to a Square; which Caſe, that it may be referred to Rule 3. we have demonſtrated (249.) I ſuppoſe

$$y = z - b$$

$$yy = z^2 - 2zb + bb$$

Therefore,

$$aa + 2zb - zz = xx$$

By Rule (247.) if v is taken at pleaſure,

$$x = vz - a$$

$$aa + 2zb - zz = vvzz - 2avz + aa$$

$$2b - z = vvz - 2av$$

$$\frac{2b + 2av}{vv + 1} = z$$

$$y = \frac{b + 2av - bvv}{vv + 1}$$

$$x = \frac{2bv + avv - a}{vv + 1}$$

PRO-

PROBLEM XXIV.

255. *To find a Number that being ſubſtracted from two given Numbers, the Remainder of each ſhall be a Square.*

Let the given Numbers be a, b, the Number ſought x, and the Squares remaining yy, zz.

$$\begin{array}{ll} a - x = yy & \quad x = a - yy \\ \underline{b - x = zz} & \\ b - a + yy = zz & \end{array}$$

Therefore $b - a + yy$ ought to be equal to a Square; and any Number v being taken at pleaſure, by Rule 2. (246.)

$$v - y = z$$

$$b - a + yy = vv - 2vy + yy$$

$$b - a = vv - 2vy$$

$$y = \tfrac{1}{2}v + \frac{a - b}{2v}$$

$$x = \tfrac{1}{2}a + \tfrac{1}{2}b - \tfrac{1}{4}vv - \frac{\overline{a - b}^{2}}{\overline{2v}^{2}}$$

PROBLEM XXV.

256. *To find two Numbers, either of which being added to the Square of the other, the Sum ſhall be a Square.*

Let the Numbers ſought be x and y, and the Squares mentioned uu and zz.

$$xx + y = uu$$

$$yy + x = zz.$$

In the firſt Equation $xx + y$ muſt be made equal to a Square, and then the Value of the Quantity x muſt be ſubſtituted in the ſecond Equation for x.

If t is taken at pleaſure,

$u =$

$$u = t - x$$
$$xx + y = tt - 2tx + xx$$
$$x = \frac{tt - y}{2t}$$
$$yy + x = yy + \frac{tt - y}{2t} = zz$$
$$4ttyy + 2t^3 - 2ty = 4ttzz \quad (250.)$$

Let there be taken s at pleaſure, and by Rule 3. (247.)

$$2tz = 2ty - s$$
$$4ttyy + 2t^3 - 2ty = 4ttyy - 4tsy + ss$$
$$2t^3 - 2ty = 4tsy + ss$$
$$y = \frac{ss - 2t^3}{4ts - 2t}$$
$$x = \frac{4t^3 s - ss}{8tts - 4tt}$$

PROBLEM XXVI.

257. *To find three Numbers ſuch, that their Sum ſhall be a Square, and alſo any two of them added together, ſhall make a Square.*

Let the Numbers ſought be x, y, z, and the Squares mentioned in the Problem rr, ss, tt, vv.

$$\begin{array}{ll}
x + y = rr & y = rr - x \quad (245.) \\
x + z = ss & \\
y + z = tt & \\
x + y + z = vv & \\
\hline
x + z = ss & x = ss - z \quad (245.) \\
rr - x + z = tt & \\
rr + z = vv & \\
\hline
rr - ss + 2z = tt & \\
rr + z = vv & z = vv - rr. \quad (245.)
\end{array}$$

I

I chuſe the ſecond of theſe Equations, becauſe it is more ſimple, and the firſt is chang'd into this:

$$2vv - rr - ss = tt.$$

Now $2vv - rr - ss$ ought to be equal to a Square, which may be referred to the ſecond Rule, if by what is explain'd at *n.* 249. we take away vv by putting

$$r = q - v$$

$$2vv - rr - ss = vv - qq + 2qv - ss = tt.$$

Now as v is an indetermin'd Number, taking q at pleaſure, I ſeek v; which Operation belongs to Rule 2. (246.) Aſſuming therefore p, I put $t = p - v$.

$$vv - qq + 2qv - ss = pp - 2pv + vv$$

$$v = \frac{pp + qq + ss}{2q + 2p}$$

p, q, s being determin'd at pleaſure, there is given v, and thence r; for $r = q - v$: therefore x, y, z, are diſcover'd; for we have

$$z = vv - rr$$

$$x = ss - z$$

$$y = rr - x$$

or

$$z = vv - rr$$

$$x = ss + rr - vv$$

$$y = vv - ss$$

PROBLEM XXVII.

258. *To find three Squares in continued Arithmetic Proportion.*

Let xx be the firſt Square, y the Difference of the neighbouring Terms in the Progreſſion:

$$xx$$

$$xx + y$$

$$xx + 2y$$

are the three Squares; that is, the two laſt are to be equal to Squares: Let theſe be uu, zz.

$$\begin{array}{ll} xx + y = uu & \quad uu - xx = y \\ xx + 2y = zz & \\ \hline 2uu - xx = zz & \end{array}$$

This Equation may be referred to the third Rule (249) if we put

$$x = t - u$$

$$2uu - xx = uu - tt + 2tu = zz$$

In ſeeking t from this Equation, we may apply the third Rule; but becauſe u is alſo indetermin'd, I refer the Quantity to Rule 2. (246.) and put

$$z = u - s$$

$$uu - tt + 2tu = ss - 2su + uu$$

$$u = \frac{ss + tt}{2s + 2t}.$$

If now s and t are taken at pleaſure, u will be given; and therefore x, for $x = t - u$, which being given, the Roots of the three Squares are known, that is x, u, $s - u$, or

$$\frac{2st + tt - ss}{2s + 2t}, \quad \frac{ss + tt}{2s + 2t}, \quad \frac{2st + ss - tt}{2s + 2t}$$

The common Denominator may in this caſe be neglected.

In this Solution I have demonſtrated that this Problem may be reſolv'd without any particular Artifice by the Rules laid down before; but the Operation may be diminiſhed, aſſuming theſe Quantities which are in Arithmetic Progreſſion.

$$tt + uu - 2tu$$
$$tt + uu$$
$$tt + uu + 2tu$$

The firſt and laſt are Squares, of which the Roots are $t - u$ and $t + u$; there is only the middle Quantity then to be made equal to a Square, which belongs to Rule 2. (246) Therefore

$$tt + uu = ss - 2su + uu$$

$$u = \frac{ss - tt}{2s}$$

Taking s and t at pleaſure, u is given, and the three Roots are known, $t - u$, $s - u$, $t + v$.

$$\frac{2st + tt - ss}{2s}, \quad \frac{ss + tt}{2s}, \quad \frac{2st + ss - tt}{2s}$$

CHAP. XIV.

Of Geometric Problems, and their Conſtruction.

259. IN the Solution of Geometric Problems, the Figure is firſt to be drawn, in which the Problem is ſuppoſed to be ſolv'd. In this Scheme, Lines are to be drawn, by which there may be form'd as much as poſſible either ſimilar Triangles or Rectangles, or other Figures whoſe Properties we know.

260. Lines muſt be deſign'd by Letters, and the known diſtinguiſhed from the unknown (13.) Their Relations are to be expreſſed from the known Properties of Figures, and from theſe, Equations are to be form'd afterwards: and then the Operations are the ſame as in other Problems.

261. The Solutions of Geometric Problems are either Arithmetical or Geometrical.

262. The Solutions are Arithmetic when the Relation of known Lines are expreſſed by Numbers,

bers, and the Solution gives the Value of the unknown, alſo expreſſed by Numbers; the Computation in this caſe does not differ from what we have ſeen above.

263. But in Geometric Solutions, the known Lines are only given, as drawn in a Plane, and the unknown are to be drawn and determin'd in the ſame Plain; and this Determination is called the Conſtruction of a Problem.

264. Lines, like other Quantities, are affirmative and negative.

Let AB be a Line whoſe beginning is A, that is, which muſt be increas'd or diminiſhed in the Extremity B. Let BC greater than AB be ſubtracted, AC will be the Exceſs; but in ſubtracting a greater Quantity from a leſſer, the Exceſs is negative; therefore AC is a negative Line: and in the indefinite Line DE, the Point A ſeparates the affirmative Lines that fall towards D, from the negative ones that are taken towards E. Tab. I. fig. 2.

265. By a like Method we demonſtrate that a Line may be taken for the Original or Limit, and all that are drawn to one part of it will be affirmative, and the others negative.

If AB is this original Line which divides the affirmative Superficies towards D from the negative one towards E; the Parts CG, CI, CL, NO, QR, of the Lines FCG, HCI, KCL, MNO, PQR are affirmative; but CF, CH, CK, NM, QP, are negative. Tab. I. fig. 3.

266. The Conſtruction of a Problem is the Conſtruction of the ſingle Equation that ſolves the Problem.

But that theſe Equations may be conſtructed, ſome of the ſimpler Problems in *Euclid's* Elements are to be made uſe of, ſuch as theſe.

IN THE FIRST BOOK.

267. Prop. 9. *To cut a given Rectilinear Angle into two equal Parts.*

268. Prop. 11. *To draw a Perpendicular to a given Line from a given Point in it.*

269. Prop. 12. *To do the ſame, if the given Point is out of the Line.*

270. Prop. 23. *To make a rectilineal Angle equal to a given rectilineal Angle.*

271. Prop. 31. *To draw a Parallel to a given Line through a Point given.*

272. Prop. 46. *To make a Square upon a given Side.*

IN THE THIRD BOOK.

273. Prop. 1. *To find the Center of a given Circle.*

274. Prop. 17. *From a given Point to draw a Tangent to a given Circle.*

IN THE SIXTH BOOK.

275. Prop. 10. *To cut a given Line in the ſame Proportion that another is cut in.*

276. *By this Propoſition a Line may be divided into any Number of equal Parts,* by taking the ſame Number of equal Parts at pleaſure on another Line.

277. Prop. 11. *Given two Lines, to find a third proportional.*

278. Prop. 12. *Given three Lines, to find a fourth proportional.*

279. Prop. 13. *Two Lines being given, to find a mean proportional.*

To theſe Problems the following ones are to be added.

280. PROBL. *To find the Side of a Square that ſhall be equal to two given Squares.*

The Sides of the two Squares given are to be join'd at right Angles (268), that there may be form'd a right-angled Triangle, whoſe Hypothenuſe is the Side ſought. *See Prop.* 47. *lib.* 1. *El.*

TAB. I. fig. 4. The Square of the Line AB is equal to the Sum of the Squares AC and BC.

281. PROBL. *To find the Side of a Square that ſhall be equal to the Difference of two given Squares.*

TAB. I. fig. 5. Let the given Sides be AB and DE, the Side of the greater Square is to be cut in half in C (276) that a Semicircle may be deſcribed upon it as a Diameter. From B there is to be applied in the Semicircle the Chord BF equal to DE, then will FA be the Line ſought. *See Prop.* 31. *lib.* 3. and alſo *Prop.* 47. *lib.* 1. *Elem.*

By the help of the Problems above-mention'd we may very eaſily conſtruct Equations of one Dimenſion, and alſo of two, after they are ſolv'd, (217).

But as the Solution of Equations of two Dimenſions is not neceſſary to their Conſtruction, I will here ſhow how each Form is conſtructed.

282. Let there be given this Equation;

$$xx + ax - bb = 0$$

or

$$xx - ax - bb = 0,$$

TAB. I. fig. 6. Let $AB = b$; let there be erected in A a Perpendicular $AC = \frac{1}{2}a$, with the Center C and Radius CA deſcribe a Circle: then if CB is drawn, BE and BD will be the Roots ſought, one of which is negative (215). In the firſt Caſe the Negative is BE, in the ſecond BD.

DEMONSTRATION.

We ſay in the firſt Caſe $BE = -x$, therefore $BC = -x - \frac{1}{2}a$, whoſe Square is equal to the

Squares of CA and AB; that is, $\frac{1}{4}aa + bb$: therefore $xx + ax + \frac{1}{4}aa = \frac{1}{4}aa + bb$; whence $xx + ax - bb = 0$.

In the ſame Caſe we ſay BD $= x$, therefore CB $= x + \frac{1}{2}a$; and again $xx + ax + \frac{1}{4}aa = \frac{1}{4}aa + bb$, that is, $xx + ax - bb = 0$.

If we put BD $= -x$, and EB $= x$, we ſhall by a like Reaſoning in each Caſe arrive at this Form $xx - ax - bb = 0$.

283. Let there be now given

$$xx + ax + bb = 0$$

or,

$$xx - ax + bb = 0$$

Let AB $= b$, and AC perpendicular and $= \frac{1}{2}a$, with the Center C and Radius CA deſcribe a Circle: If BE is erected perpendicular to BA, cutting the Circle in D and E, then will BD and BE be the Roots ſought, which in the firſt of theſe Forms are both negative, and in the ſecond both affirmative (215).

Tab. I. fig. 7.

DEMONSTRATION.

Draw D*d*, E*e* parallel to BA, then A*e* and BE are equal, as alſo A*d* and BD; and A*e*, *e*E, *e*F, and A*d*, *d*D, *d*F are proportional: that is, for the firſt form ∻ $-x$, b, $a + x$; and for the ſecond ∻ x, b, $a - x$. Whence the very Equations of thoſe Forms are produced.

CHAP. XV.

Geometrical Problems.

PROBLEM XXVIII.

TAB. I. fig. 8. 284. *TO inſcribe a Square in a given Triangle.*

Let the given Triangle be ACB, and the Square to be deſcribed EFHG; and from the Vertex C to AB let there be drawn the Perpendicular CD.

Let now $AD = a$
$AB = b$
$CD = d$
$AF = y$
$FE = x$

In the ſimilar Triangles AFE, ADC, AF (y), FE (x), :: AD (a), DC (d).

$$ax = yd$$

In the ſimilar Triangles BHG, BDC, BH = BA — AF — FE ($b - x - y$) HG = FE (x) :: BD ($b - a$), DC (d).

$$bx - ax = db - dy - dx.$$

Putting for dy its Value ax, we have

$$bx + dx = db$$

$$b + d,\ b :: d,\ x \cdot (135).$$

x being known, if in the Perpendicular DC you take DI equal to it, and through the Point I, draw a Parallel to the Baſe; this Parallel will determine the Points E and G.

PRO-

PROBLEM XXIX.

TAB. I. fig. 9.

285. *Let* AB *be the Diameter of a Semicircle whoſe Center is* C; *and let* D, E *be two Points equidiſtant from* C: *The Point is ſought in the Circumference, from which if the Lines* FD *and* FE *are drawn, the Radius may be a Mean Proportional betwixt the Sum and Difference of thoſe Lines.*

Let FG be drawn perpendicular to the Diameter, and let DC = CE = a

$$FC = b$$
$$FD = x$$
$$FE = y$$
$$CG = z$$

The Condition of the Problem is this:

$$\div\, x + y,\ b,\ x - y.$$

Therefore,

$$1^\circ.\ bb = xx - yy$$

in the Triangle FDC, by *lib.* II. *Prop.* 11. *El.*

$$2^\circ.\ xx = aa + bb + 2az.$$

in the Triangle FCE, by *lib.* II. *Prop.* 12. *El.*

$$3^\circ.\ yy = aa + bb - 2az.$$

This third Equation being ſubtracted from the ſecond, gives

$$xx - yy = 4az.$$

Whence in the firſt Equation we find

$$bb = 4az$$
$$\div\, 4a,\ b,\ z.$$

Now z being known (277) the Point G is given; whence a Perpendicular to AB being drawn, the Point F is determin'd.

OBSERVATION VI.

286. In this Problem we ſeek z, not x or y; becauſe z has but one Value, but x or y more, as x for inſtance. This is the greater of the

two Lines, and tho' there can but one ſuch Line be drawn from the Point D to the Semicircle AFB, yet there may be another equal to it drawn in the oppoſite Semicircle; for the Lines D*f*, E*f* ſatisfy the Problem. But theſe Lines are negative (265), therefore x has two equal Values DF, D*f*, one of which is affirmative, and the other negative: and therefore x cannot be found but in an Equation that contains the Value of the Square xx. This Equation may be diſcover'd by adding the three Equations.

$$\begin{array}{l} xx - yy = bb \\ xx = aa + bb + 2az \\ yy = aa + bb - 2az \\ \hline 2xx = 2aa + 3bb \\ xx = aa + \frac{3}{2}bb \end{array}$$

287. It may be univerſally, that when there is a Line that has the ſame relation with reſpect to two other Lines that ſolve the Problem; as here DG, to DF and D*f*; then this Line is to be ſought, if the determining it gives the Solution of the Problem.

PROBLEM XXX.

TAB. II. fig. 1.

288. *The ſame things being given to diſcover the Point* F *ſuch, that the Radius may be a Mean Proportional betwixt the Lines* FD, FE.

The Lines being deſign'd with the ſame Letters as in the precedent Problem, we have theſe Equations.

$$\begin{array}{l} xy = bb \\ xx = aa + bb + 2az \\ yy = aa + bb - 2az \end{array}$$

In theſe Equations there is nothing that confines x to expreſs the greater Line rather than the leſſer; it muſt therefore expreſs both: for in the two laſt Equations y will be the greater, if z is negative; conſequently, x or y will have two affirmative Values. But they have alſo in the oppoſite Semicircle two correſponding negative Values, therefore the Squares xx and yy will have two Values. If we ſeek x then, we ſhall neceſſarily have an Equation of four Dimenſions; which, if we ſeek the Value of the Square xx, is reſolv'd as an Equation of two Dimenſions, becauſe of the two Values of this Square.

But if z is determin'd, the Problem is alſo ſolv'd; becauſe G being given, F is determin'd: But z has only two Values, one affirmative CG, and the other negative Cg; I ſeek therefore z, becauſe I ſhall have an Equation that will give the Value of the Square zz.

The firſt Equation gives

$$xx = \frac{b^4}{yy}$$

The ſecond is chang'd into this,

$$\frac{b^4}{yy} = aa + bb + 2az.$$

And making uſe of the third Equation,

$$yy = \frac{b^4}{aa + bb + 2az} = aa + bb - 2az$$

Which laſt Equation, when reduced, (171, & *ſeq.*) is chang'd into this;

$$zz = \tfrac{1}{4}aa + \tfrac{1}{2}bb$$

Let the Square $\frac{1}{4}bb$, whoſe Root is $\frac{1}{2}b$, be doubled (280); then we have the Root of the Square $\frac{1}{2}bb$: and ſeeking a Square equal to two Squares (280) the Problem is conſtructed.

Problem XXXI.

Tab. II. fig. 2. 289. Given a Square ABCD, to draw a right Line from the Angle A, ſo that the part FE intercepted betwixt the Side CD and the Side BC continued, may be equal to a given Line.

By conſidering this Problem, without any regard to the Reſtrictions that cannot be conſider'd in the Solution, it will be evident that the Problem may be reduced to this: From the Point A to draw a Line ſo that its Part intercepted in the Angle form'd by the Lines DC, BC may be equal to a given Line.

And we may eaſily ſee that the Problem has four Solutions; for the Lines AFE, A*EF*, *e*A*f*, eAf ſatisfy the Problem; for each of the Lines FE, *EF*, *ef*, ef, is equal to the given Line.

Now if we ſeek AF, the Equation will alſo give *AF*, Af, and A*f*, and will be an Equation of four Dimenſions.

If we ſeek DF, we diſcover beſides *DF*, Df, D*f*.

But we muſt obſerve, that FE, *FE* are ſimilarly poſited, as alſo *fe*, fe; and therefore different Lines are given, affected in the ſame manner with reſpect to FE and *FE*. Now ſuch an one is to be choſen from amongſt theſe, as will, when known, ſolve the Problem (287).

If we conceive the middle Points LL of the Lines FE, *FE* to be join'd, the Line joining them will cut the Diagonal of the Square AC produced in M, and AM or CM is poſited in the ſame manner with reſpect of either of the Lines. But the Point M being given, the Problem is ſolv'd; becauſe ML is perpendicular to AC, and CL is equal to LE which is given.

Therefore AM or CM is to be ſought, either of which has only two Values AM, A*m*, or CM, C*m*.

Examining the Figure, I ſee the Computation will be ſimpler if CM is ſought. This therefore I call x, and put

$$\begin{aligned} \mathrm{CM} &= x \\ \mathrm{FE} &= 2a \\ \mathrm{AB} = \mathrm{AD} &= b \\ \mathrm{AC} &= c \\ \mathrm{AF} &= y \\ \mathrm{FD} &= z \end{aligned}$$

The Triangles ABE, AFD are ſimilar, and AE $(y + 2a)$ AB (b) :: AF (y), FD (z).

In the right-angled Triangle AFD, the Square AF (yy) is equal to the Squares $zz + bb$.

Again, in the Triangle ACL becauſe LM is perpendicular to AC, the Square AL $\overline{y + a}^2$ is equal to the Squares of the Sides AC, CL, and twice the Rectangle ACM; that is, $cc + aa + 2xc$. There are given therefore for the three unknown Quantities, theſe Equations:

$$1^\circ.\ by = zy + 2za, \text{ or } z = \frac{by}{y + 2a}$$

$$2^\circ.\ yy = zz + bb$$

$$3^\circ.\ yy + 2ya = cc + 2xc.$$

By ſubſtituting the Value of z, the ſecond Equation is chang'd into this:

$$yy = \frac{bbyy}{yy + 4ay + 4aa} + bb$$

$$y^4 + 4ay^3 + 4aayy = 2bbyy + 4bbay + 4bbaa.$$

The firſt Member of this is the Square of the firſt Member of the third Equation, whence we deduce a new Equation:

$$c^4 + 4c^3x + 4ccxx = 2bbyy + 4bbay + 4bbaa.$$

By reaſon of the right-angled Triangle ABC,

$$cc = 2bb.$$

Therefore

$$c^4 + 4c^3x + 4ccxx = ccyy + 2ccay - 2ccaa$$

$$cc + 4cx + 4xx = yy + 2ay + 2aa.$$

For $yy + 2ay$, I put the Value of it that is given in the third Equation, and we have

$$xx + \tfrac{1}{2}cx - \tfrac{1}{2}aa = 0.$$

The Conſtruction of which is eaſy (282).

OBSERVATION VII.

290. But the Number of Geometric Solutions does not always determine the Number of Dimenſions of the Equation by which the Problem is ſolv'd. The Dimenſions of this Equation oftimes exceeds the Number of poſſible Solutions; which is owing to three Cauſes.

When the Conditions of the Problem algebraically expreſſed have algebraic Solutions, which cannot be applied to the Problem geometrically conſider'd.

There are beſides two Cauſes explain'd before (220, 223, 235) but it may be obſerv'd, that what was explain'd in *n.* 220. cannot be applied to Geometric Problems, unleſs the Problem does not permit an affirmative Root to be alſo negative, and *vice verſa*: for in Geometry the Affirmation or Negation does frequently depend on the Figure which may be drawn otherwiſe. The two following Problems will illuſtrate theſe three Caſes by Examples.

PROBLEM XXXII.

291. *To form a Square, the Difference betwixt the Diagonal and the Side being given.*

Let the Square be DCEF, let the Side be called x, and the given Difference AB $= a$; the Square of the Diagonal DE $= x + a^{2}$ is equal to the Squares of DC and CE, that is, $2xx$. Therefore TAB. II. fig. 3.

$$xx + 2ax + aa = 2xx$$
$$xx - 2ax - aa = 0.$$

This Equation has two Roots, but the affirmative only ſolves the Problem: the negative one is equal to the Side taking away $2a$; for if this Difference is called $-x$, we come to the ſame Equation $xx - 2ax - aa = 0$. But this is not at all intereſted in the Problem geometrically conſidered, as is evident from the Figure itſelf.

PROBLEM XXXIII.

292. *There is given a Semicircle* AEB, *and to* AB *continued there is an indetermin'd Perpendicular* DF, AF *is to be drawn ſo that* $\div$ EF, DF, AE. TAB. II. fig. 4.

It is evident at firſt view, that FD may be continued below AD, as alſo the Sermicircle continued to a Circle, and that none of the Lines EF, DF, AF can be diſcover'd, unleſs we alſo find an equal negative Value at the ſame time. The Equation will therefore neceſſarily contain the Square of the Quantity ſought.

EF $= x$
DF $= y$
AE $= z$
AB $= a$
AD $= b$

The Equations are deduced from theſe Proportions:

$$\div \; EF\ (x),\ DF\ (y),\ AE\ (z)$$

$$AB\ (a),\ AE\ (z),\ ::\ AF\ (z+x),\ AD\ (b)$$

$$AF^q\ \overline{z+x}^2 = FD^q\ (yy) + AD^q\ (bb).$$

Therefore,

$$1.\quad xz = yy.$$

$$2.\quad zz + zx = ba.$$

$$3.\quad zz + 2zx + xx = yy + bb.$$

The third Equation, after the firſt and ſecond are ſubſtracted from it, is reduced to this:

$$xx = bb - ba.$$

But although x is now given, the Problem is not yet ſolv'd; for I cannot draw AF, unleſs AE or DF is given, that is, z or y. I ſeek z, becauſe it is eaſieſt to be diſcover'd.

The ſecond Equation is changed into this;

$$zx = ba - zz$$

$$zzxx = bbaa - 2bazz + z^4$$

Therefore,

$$zzbb - zzba = bbaa - 2bazz + z^4$$

That is,

$$\begin{aligned} z^4 - abzz + bbaa &= 0 \\ - bbzz \quad & \end{aligned}$$

Which Equation has four Roots, two poſitive and two negative ones, equal to them; that is, the Square zz has two Values, but one of them is to be rejected (230), becauſe z^4 is the Square of the Root $-zz$, not of $+zz$; and this follows from the Nature of the Problem, and not from a Line drawn at pleaſure.

We may alſo reduce this Equation to two Dimenſions, if we put

$$bb - ba = dd.$$

Then x is equal to $+d$ and $-d$, and ſubſtituting theſe Values of x in the Equation $zx = ba - zz$, we have theſe two Equations;

$$zz + dz - ba = 0$$
$$zz - dz - ba = 0.$$

Each of which Equations has one affirmative and one negative Root.

I reject both the Roots of the laſt Equation, becauſe we have nothing to do with the negative Value of x.

In the firſt, I chuſe the poſitive, and reject the negative Root.

293. It is to be obſerv'd, that this negative Root, with the affirmative one of the Equation $zz - zx - ba = 0$, belong to impoſſible Caſes (235, 236). For if we ſeek y, we ſhall ſee that the Problem has two Solutions; which may indeed be algebraically expreſſed, but are in reality impoſſible.

The firſt Equation is chang'd into this:

$$y^4 = zzxx = bbzz - bazz$$

The ſecond is chang'd into this,

$$zz = ba - yy.$$

Therefore

$$y^4 = \overline{bb - ba} \times \overline{ba - yy}$$

Or,

$$\begin{aligned} y^4 + bbyy - ba^3 &= 0 \\ - bayy + bbaa. & \end{aligned}$$

This Equation gives two Values of the Square yy, one of which is negative; and conſequently y has two impoſſible Values (218).

OBSERVATION VIII.

294. Sometimes it happens, that the Number of Solutions exceeds the Number of Dimenſions of the Equation by which the Problem is ſolv'd; which is a Paradox: ſince from what has been explain'd before (212), it is manifeſt that the unknown Quantity cannot have more Values than it has Dimenſions in the ſingle Equation.

This Caſe has place only in Geometry, and may ſometimes happen becauſe the Number of Geometric Solutions may exceed the Number of Algebraic Solutions. This falls out when many equal affirmative or negative Lines ſolve the Problem, for various equal Values are algebraically the ſame Value.

295. But it is to be remark'd, that on ſuch occaſions we have not always a ſimpler Equation; for an Equation may contain two equal Values of the ſame unknown Quantity: that is, conſidering the thing algebraically, twice the ſame Value. In this Equation, $xx - 2ax + aa = 0$, x has two Values equal to a; which, except in a Geometric Problem, ſignifies nothing but that x is equal to a: for if $x = a$, we have this Equation, $x - a = 0$, or $xx - 2xa + aa = 0$, or even more compounded. But we do not always arrive at that which expreſſes the Number of equal Geometric Solutions. This Obſervation is illuſtrated by the following Problem.

PROBLEM XXXIV.

TAB. II. fig. 5. 296. *In the Circumference of the Semicircle* ADB, *whoſe Center is* C, *there is ſought the Point* D *ſuch, that if from it to the Diameter* AB, *there is drawn the Perpendicular* DE, *the Rectangle under* CE *and*

ED *may be equal to the Square of the given Line* FH.

Let us put

$$CD = a$$
$$CE = x$$
$$DE = y$$
$$FH = b$$

And we have theſe Equations:

$$xy = bb \qquad y = \frac{bb}{x}$$
$$aa = xx + yy.$$

And by Subſtitution,

$$aa = xx + \frac{b^4}{xx}$$
$$x^4 - aaxx + b^4 = 0.$$

Which Equation, if ſolv'd (217), gives two Values of the Square xx; which have each a poſitive Root and an equal negative one: x (CE) therefore has two Values on each ſide the Center; and four Points E, E, e, e, may be determin'd by this Solution.

Each of theſe Points afford two Solutions, one affirmative above the Line AB, and the other negative below the Line; y therefore has four affirmative, and as many negative Values. But in the given Equations if we ſeek y, we have

$$y^4 - aayy + b^4 = 0.$$

Which Equation gives only two poſitive and as many negative Values: The reaſon of this is manifeſt, becauſe of the Equality of ED, eD, and ED, eD (294, 295).

297. But that this Equation $x^4 + aaxx + b^4 = 0$ may be conſtructed, it muſt firſt be ſolv'd (217),

(217), or reduced to two others more ſimple; and we have

$$xx = \tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 - b^4}$$
$$xx = \tfrac{1}{2}aa - \sqrt{\tfrac{1}{4}a^4 - b^4}.$$

I ſeek a third Proportional to the Lines a and b (277), which I call d; then

$bb = ad$, and $b^4 = aadd$.

Whence $\sqrt{\tfrac{1}{4}a^4 - b^4} = a\sqrt{\tfrac{1}{4}aa - dd}$ (194.)

Now $\sqrt{\tfrac{1}{4}aa - dd}$ may be determin'd (281). I put

$$\sqrt{\tfrac{1}{4}aa - dd} = e.$$

And making Subſtitution, we have

$$xx = \tfrac{1}{2}aa + ae$$
$$xx = \tfrac{1}{2}aa - ae.$$

The Conſtruction of either of which is not difficult (279).

We might refer hither *n.* 199. See PROB. II. which does not differ from this.

OBSERVATION IX.

298. We ſee how in Geometric Problems negative Solutions are ſometimes rejected: This happens for the moſt part when we ſeek a Conſtruction on one ſide of a Line, which takes place alſo in the ſame manner on the other ſide. As to the reſt, Roots are rarely rejected in Geometry, becauſe they are negative. For a negative Line differs only in Poſition from a poſitive one (264, 265.) Wherefore oftimes poſitive Solutions are chang'd into negative ones, and *vice verſa* only by varying the known Quantities; and various Caſes may be determin'd, only by con-

ſidering the Equation by which the Problem is ſolv'd: Which we will illuſtrate by an Example.

PROBLEM XXXV.

299. *To draw a Circle which may paſs through a given Point, and may touch a Line given in Poſition; as alſo a Circle given in Magnitude and Poſition in the ſame Plain.*

Let A be the Point given, B D the indetermined Line given, E F the Circle given, whoſe Center is C. TAB. III. fig. 1.

The Center of the Circle ſought I ſuppoſe to be N; drawing N C, E will be the Point of Contact of the given and ſought Circles; drawing N I perpendicular to B D, I will be the Point of Contact of the Circle ſought, and the given Line: NA, NI, and N E, are therefore Radius's of the ſame Circle, and conſequently equal.

I draw AB perpendicular to BD, which I divide into two equal Parts at M, and draw M L parallel to BD; I ſuppoſe AI drawn, which cuts ML in O, and alſo in two equal Parts, becauſe of the Equals AM and MB; therefore in the Iſoſceles Triangle ANI, NO is perpendicular to AI, and NOI is a right-angled Triangle, in which OL is perpendicular to the Hypothenuſe, and

$$\div \text{NL, LO, LI.}$$

If NI is ſuppos'd produc'd to H, ſo that IH may be equal to EC, and ſuppoſing alſo CH drawn, the Triangle CNH will be Iſoſceles. Let there be drawn now CD perpendicular to BD, and let this be produc'd to G, that DG and IH or EC may be equal, that is, HG parallel to BD; let CG be cut in Q, in two equal Parts, and drawing QR parallel to BD, it will cut CH equally in P, and NP will be perpendicular to HC, wherefore the Triangle NPH is right-angled, and PK is perpendicular to its Hypothenuſe, whence it follows

∺ NK, KP, KH.

Let us put

$$\begin{aligned} BD &= 2a \\ CQ = QG = KH &= b \\ AM = MB = LI &= c \\ MR = LK &= d \\ MO = OL &= x \\ NL &= y \end{aligned}$$

Therefore $PQ = KP = a - x$

We have only two unknown Quantities x and y, which being given, N may be determin'd; we have alſo two continued Proportions, which algebraically expreſſed, are

∺ y, x, c.

∺ $d + y$, $a - x$, b.

Theſe give the Equations

$$xx = yc \text{ or } y = \frac{xx}{c}$$

$$aa - 2ax + xx = db + by$$

And by Subſtitution,

$$aa - 2ax + xx = db + \frac{bxx}{c}$$

Whence

$$xx - \frac{2cax}{c-b} + \frac{aac - dbc}{c-b} = 0$$

CASE I.

300. If c exceeds b, that is, if AB is greater than CG, the ſecond Term of the Equation is negative; if in the ſame Caſe aa exceeds db, that is, if BD exceeds double the mean Proportion

betwixt MR and CQ, the laſt Term will be affirmative, and the Equation has two affirmative Roots (215); that is, two Circles have the requiſite Properties, both whoſe Centers are towards the ſame part of the Line AB with CD, but one of their Centers lies betwixt theſe Lines, the other beyond CD.

CASE II.

301. If c remaining greater than b, the Rectangle db exceeds aa, the third Term will be negative, and one Root is negative, the other affirmative (215); that is, the Line AB will be given betwixt the two Centers of the Circles that ſolve the Problem.

CASE III.

302. This alſo happens if b exceeds c, and aa exceeds db (215); for then the ſecond Term is affirmative, and the third negative.

CASE IV.

303. But if when b exceeds c, the Rectangle db exceeds aa, all the Terms are affirmative, and both the Roots negative (215); wherefore both the Centers are on that Side AB, which is betwixt it and the Line C Q.

CASE V.

304. If $aa=bc$, the Equation is chang'd into this,

$xx-\frac{2cax}{c-b}=0$, whoſe Roots are

$$x=0$$

$$x=\frac{2ca}{c-b}$$

That is, one Center lies in the Line AB, and the other lies in the ſame part of this Line with CD, if c exceeds b; but otherwiſe, if b exceeds c.

CASE VI.

305. If $c=b$ the Equation multiplied by $c-b$ is

$$xx \times \overline{c-b} - 2acx + aac - dbc = 0$$

But $c-b=0$; therefore we have

$$2ax = aa - db$$

$$x = \frac{aa-db}{2a}$$

And the Problem has only one Solution, the other Center going off *in infinitum*.

306. But as to the Conſtruction of this Equation

$$xx - \frac{2acx}{c-b} + \frac{aac-bdc}{c-b} = 0,$$

we muſt ſeek a fourth Proportional to theſe three $c-b$, c, a, which we may call f; and another to theſe three a, b, d, which I call g;

Therefore $c-b,\ c :: a,\ f, = \frac{ac}{c-b}$

$a,\ b :: d,\ g$, and $ag=bd$.

The Equation then is changed into this

$$xx - 2fx + \overline{a-g} \times f = 0$$

Now finding a mean Proportional betwixt $\overline{a-g}$ and f, and calling it h, we have this Equation

$$xx - 2fx + hh = 0$$

The Conſtruction of which we have already explained (217).

When b exceeds c, it is $+2fx$, but on the contrary $-2fx$ when c exceeds b. When a is greater than g, we have $+hh$, but $-hh$, when g is greater than a.

OBSERVATION X.

307. In Geometric Problems it is not always neceſſary to continue the Computation 'till we come to a ſingle Equation. For when we have an Equation that gives the Proportion betwixt two unknown Quantities, if only thoſe two remain, the Figure is to be examin'd, becauſe oft-times from ſuch an Equation we deduce a Conſtruction; as will be evident in the following Problem, and in Prob. 41.

PROBLEM XXXVI.

308. *To deſcribe a Circle that may touch another Circle given in Magnitude and Poſition, and may alſo paſs through two Points given in the ſame Plane with the Circle.*

Let the given Points be A and B, the Circle given DD, whoſe Center is C. Tab. III. Fig. 2.

Draw AB, and let it be biſected in F, and in that Point let FG be erected perpendicular to it.

It is evident that the Center of the Circle ſought is in this Perpendicular, ſuppoſe it to be N; and let us conceive the Lines NA, NC, drawn, of which this laſt cuts the given Circle in E; NA, NE, muſt be equal.

From C let fall the Perpendicular CH on FG, and let

$$FH=a$$
$$FA=b$$
$$CH=c$$
$$CE=d$$
$$NH=x$$
$$NE=NA=y$$

In the right-angled Triangles NCH and NFA, we have

$NC^q = CH^q + NH^q$, and $NA^q = AF^q + FN^q$; That is

$$yy + 2dy + dd = cc + xx$$
$$yy = bb + aa + 2ax + xx$$

I ſubſtract the ſecond Equation from the firſt.

$$2dy + dd = cc - bb - aa - 2ax$$
$$2dy = cc - bb - aa - dd - 2ax$$

I ſeek a Square equal to $cc - bb - aa - dd$ (280, 281); let this be ee.

I then ſeek a third Proportional to theſe two Quantities, $2a$, e, which I call f; and

$$2af = cc - bb - aa - dd.$$

Therefore

$$dy = af - ax.$$

Whence

$$y, f - x :: a, d.$$

If now $HI = f$, NI is equal to $f - x$, and it will be

$$NE, NI :: a, d.$$

If we conceive I E drawn, and CL parallel to it, it will be alſo

$$a, d :: EC\ (d), IL = \frac{dd.}{a}$$

Whence we deduce this Conſtruction. Suppoſe the Lines AB, FG, CH, which are given to be drawn, as is before mentioned; I take $HI = f$ which is given, and $IL = \frac{dd}{a}$ which is alſo given. I draw LC, and through the Point I, a Parallel to it IE, which cuts the Circle DE in E and e; and through C and E, I draw a Line which produc'd cuts FG in N; which is the Center ſought, as is evident from what has been before explain'd.

The Point e ſhews that there is another Solution of the Problem, and e C produc'd, determines by its Interſection with the Line FL, the Center of the ſecond Circle.

When N falls to the other part of the Point H, x is negative, but that does not change the Conſtruction.

When $bb + aa + dd$ exceeds cc, we put $bb + aa + dd - cc = ee$, and in this caſe f is negative; wherefore the Point I is mark'd on the other Side of the Point H (264).

OBSERVATION XI.

309. Having already explained thoſe Things which are to be obſerved in algebraic Solutions of Geometric Problems, we may now add that Algebra is not always neceſſary, for in many Caſes a Conſtruction is eaſily deduc'd from the well known Properties of Figures; wherefore this ought firſt to be attempted before we enter on the Computation.

PROBLEM XXXVII.

310. *To deſcribe a Circle which may touch two given right Lines, and alſo paſs through a Point given in the ſame Plane.*

Let A be the Point given; and BD, EF the Lines given, let theſe be produc'd 'till they cut each other in G; let GN be drawn, which divides the Angle DGE into two equal Parts, (267), it is evident that the Center of the Circle ſought is in this Line. From A let fall the Perpendicular AI upon this Line (269); it is again evident, that if this is produc'd, 'till IH is equal to AI, H will be in the Circumference of the Circle ſought. TAB. II, fig. 6.

This premiſed, let us ſuppoſe N to be the Center of the Circle ſought, and NM to be perpendicular to EF, NM will be the Radius of this Circle, and M the Point of Contact.

All this is to be conſider'd before the Computation is begun, which I preſently ſee would be unneceſſary, if we continue the Line AH 'till it cuts EF in K; for it is evident, that KM is a mean Proportional betwixt the known Lines KA, KH (36. III *E.*) wherefore M is given (279); in which Point if there is a Perpendicular erected, it will cut the Line GI in N, and will determine the Center of the Circle ſought.

OBSERVATION XII.

311. There are ſometimes Conditions given in Geometric Problems, which cannot be algebraically expreſſed; in theſe Caſes, the Solution is to be deduc'd ſolely from Geometric Conſiderations.

PROBLEM XXXVIII.

312. *A Point and a Parallelogram being given in the ſame Plain, through the Point to draw a Parallel to a Line given by Poſition in the ſame Plain, without making uſe of a Circle.*

TAB. III. fig. 3. There is to be drawn a Parallel to BF, through the Point A, only by drawing ſtraight Lines, the Parallelogram IL being given in the ſame Plain.

It is evident by conſidering the Problem, that Algebra is here unuſeful; for every algebraic Solution produces an Equation, which can never be conſtructed without making uſe of a Circle. The Solution is therefore to be deduc'd from Geometric Conſiderations.

I produce three Sides of the Parallelogram, 'till they occur to the given Line in B, C, & F, which is alſo cut in E by the Diagonal KM produc'd;

duc'd; the fourth Side IK is produc'd indeterminately.

This premis'd, the Conſtruction is thus: From A, I draw Lines to B and C; the firſt of which cuts the Side IK, or its Continuation, in D, whence I draw DE, cutting AC in G, and through that Point draw FG, which occurs to the Side IK, or its Continuation in H; AH will be parallel to BF.

DEMONSTRATION.

From C to T the Interſections of the Lines FL and ED, I draw the right Line CT.

Becauſe of the Parallels KB, MC;

EC, EB : : EM, EK.

And by reaſon of the Parallels DK, NM;

ET, ED : : EM, EK.

Therefore

EC, EB : : ET, ED.

Wherefore ADB and NC are parallel; whence it follows, that the Triangles CGN and AGD are ſimilar; and

GN, GD : : GC, GA

But by reaſon of the Parallels TF, DH, the Triangles FGT and HGD, are ſimilar; and

GT, GD : : GF, GH.

Therefore

GC, GA : : GF, GH

And the Triangles CGF, AGH are ſimilar (*6. VI E.*) for the Angles FGC, HGA, being oppos'd at the Vertex, are equal; therefore the Angles CFG and AHG are equal, and AH parallel to FC (27. 1 *E.*)

Different Caſes of this Problem may be given, but the Solution of them all is eaſy from the Conſtruction here laid down, which takes place in this Figure, wherever the Point A may be ſuppos'd.

But when by reaſon of the Situation of the Parallelogram, with reſpect to the Line BE, all the Points B, C, E, F, cannot be determin'd; another Parallelogram is to be drawn, which is eaſily done when one is given.

Tab. III. fig. 4. 313. Let IKLM be the given Parallelogram, whoſe Diagonal is KM; let there be drawn two Lines RS, PQ, at pleaſure, cutting each other in O, ſome Point of the Diagonal, the firſt occurs to the Sides IK, ML of the given Parallelogram in R and S; the other Line PQ cuts the Sides in P and Q, the Lines joining R, Q and P, S, will be parallel; and being continued, as alſo the Sides KL, MI, we ſhall have a new Parallelogram TQVP.

DEMONSTRATION.

Becauſe of the Parallels KR, MS, the Triangles MOS, ROK, are ſimilar; as alſo the Triangles POM, KOQ, becauſe of the Parallels MP, KQ. Therefore

SO, OR : : MO, OK
PO, OQ : : MO, OK
Conſequently
SO, OR : : PO, OQ

Wherefore the Triangles PSO and ROQ are ſimilar, and PS, RQ parallel.

* * *

CHAP.

CHAP. XVI.

Of Phyſical Problems.

314. PHyſical Problems are ſolv'd like others, if we only remember the Rule laid down *n.* 183. to make uſe of what is known from other Principles relating to the Queſtion, in order to bring it to an Equation.

PROBLEM XXXVIII.

315. *Given the Height of the Quickſilver in the Barometer, to determine the Height of the Quickſilver in a cylindric Tube, whoſe Length is given, and which has a given Quantity of Air included in it.*

Let the given Altitude of the Mercury in the Barometer be a; the Height of another cylindric vertical Tube, above the Superficies of the Mercury, b; we ſuppoſe this to be clos'd on top, and a Quantity of Air in it, which in the State of the exterior Air, would occupy the Space c in the Tube; the Height the Mercury will be ſuſtain'd at in this Tube, by the Preſſure of the Atmoſphere, is to be determin'd. Let this Height be called x.

This Problem cannot be reſolv'd, unleſs the Law, according to which the Air expands it ſelf, is known; which is, That the Space taken up by Air is inverſely as the compreſſing Forces.

When the Air in the Tube takes up the Space c, it is compreſſed by the whole Atmoſphere, and the compreſſing Force can ſuſtain a Column of Mercury whoſe Height is a.

But the Air in the Tube is dilated, and takes up the Space $b-x$; and the Preſſure of the Atmoſphere produces two Effects, it ſuſtains the Mercury at the Height x, and by its remaining Force, capable of ſuſtaining a Column of Mercury $a-x$, compreſſes the Air in the Tube.

Therefore when the Spaces taken up by the Air are c and $b-x$, the compreſſing Forces are as a to $a-x$. Conſequently

$$c,\ b-x\ ::\ a-x,\ a$$

$$ca = ba - bx - ax + xx$$

$$\begin{array}{l} xx - ax + ba = 0 \\ \quad\;\; - bx - ca \end{array}$$

Which Equation has two poſitive Roots (215), but one only ſerves the preſent Purpoſe; it may be diſtinguiſh'd from the other (230), if we remember that xx in the Equation had the negative Root $-x$.

PROBLEM XXXIX.

316. *Given a Mixture of two known Metals, to determine how much of each Metal it contains.*

That this Problem may be ſolv'd, it is enough to know, that different Metals, equal in Weight, loſe unequal Parts of their Weights, when they are weighed in Water.

Let there be given a Mixture of Gold and Silver, let the Weight of the Mixture be a; let there be given beſides, any two Pieces, one of pure Gold, and the other of pure Silver; let theſe be weighed in Water, as alſo the Mixture.

Each of theſe Bodies loſe ſomething of their Weight; let b be the Weight loſt of the Mixture a, let the Weight of the Gold be c, and the Weight it loſes in Water be d; let the Weight of the Silver be e, and the Weight it loſes f.

Call the Weight of the Gold in the Mixture x, and the Weight of the Silver will be $a-x$; let the Weight of the Gold, in the Mixture loſt in the Water, be called y, then the loſs of Weight of the Silver in the ſame Mixture will be $b-y$.

It is evident that the Weight of the Gold c is to the Weight of the Gold x, as the Weight loſt in c to the loſs of Weight in x, which muſt alſo be applied to the Silver e and $a-x$. Therefore,

$$c,\ x :: d,\ y$$
$$e,\ a-x :: f,\ b-y$$

Whence we deduce

$$dx=cy, \text{ or } y=\frac{dx}{c}$$

$$e,\ a-x :: f,\ b-\frac{dx}{c}$$

$$fa-fx=be-\frac{dex}{c}$$

$$x=\frac{cfa-cbe}{cf-de}$$

PROBLEM XL.

317. *From a given Point to throw a Body, with a given Velocity, to a given Point* B. Tab. III. fig. 5.

When the Velocity is given, the Altitude may be determined, from whence a Body falling will acquire that Velocity: Let this be DA, which Line we will place vertically.

It is well known in Phyſics, that a Body thrown whilſt it moves with an uniform Motion, along the Line of its Projection, does, mean while, fall through Spaces, which are as the Squares of the Times, theſe two Motions not diſturbing each other. It is alſo known, that a Body, with the

Velocity acquir'd by falling from a certain Height, will in the ſame time it fell, go through a Space equal to twice the Height it fell from.

This premis'd, let AE be the Direction ſought, in which the Body by its projectile Motion, runs the Space AE in the ſame time, that in falling it goes through EB ſuppos'd vertical.

Let us put

$$AB = a$$
$$AD = b$$
$$AE = x$$
$$EB = y$$

In the time that the Body falls through y, by its projectile Motion it runs through x.

In the time the Body falls through b, by its Projectile it will run through $2b$.

The Spaces deſcrib'd by an uniform Motion, with the ſame Velocity, are as the Times; but the Spaces fell through, are as the Squares of the Times. Therefore

$$xx,\ 4bb :: y,\ b$$
$$4by = xx$$
$$\div\ 4b,\ x,\ y$$

I produce AD to F, ſo that AF = 4 AD; I now ſee that the Line AE is ſo to be drawn, that AF, AE, EB, may be in continual Proportion; which they will be, if EF being drawn, the Triangles AEF and AEB are ſimilar, or the Angles AEF, EAB equal: whence it follows, that the Problem is ſolv'd, if there is a Circle deſcrib'd through the Point F, that touches the Line AB in A, which is very eaſy. This Circle will cut the vertical Line paſſing through B (32. III *E.*) in the Point E. But as it cuts this Line in two Points, it follows that the Body may be thrown in two different Directions. See Obſerv. 10. *n.* 307.

This Conſtruction of this celebrated Problem is very ſimple and univerſal. I attain'd it by the Method here laid down, but afterwards ſaw it explain'd by Mr. *Cotes*, Profeſſor at *Cambridge.*

If in ſeeking the ſecond Equation we had compleated the Algebraic Solution, the Conſtruction of the Equation that contained the Solution of the Problem, would have been more compounded; and meer Geometric Conſiderations would, with more Difficulty, have brought us to this Conſtruction.

FINIS.

A
SPECIMEN
OF A
COMMENTARY
ON
Sir ISAAC NEWTON's
Univerſal Arithmetic.

THE PREFACE.

ALL Authors do not write meerly for the uſe of Beginners, neither can it be reaſonably demanded of them. Who could expect that Sir Iſaac Newton, *whilſt he lays open to Philoſophers, the ſublimer parts of Mathematics and Phyſics, ſhould write a Treatiſe of the firſt Rudiments of Algebra? He being formerly Mathematical Profeſſor at* Cambridge, *explain'd theſe Elements to the Youth of the Univerſity; and then delivered in the written Lectures, according to the Cuſtom of the Place, but never intended them for the Preſs: So that this Treaſure would ſtill have been hid, had it not been publiſhed contrary to the Will of the Author. Which Action we cannot indeed commend, tho' we congratulate the Lovers of the Mathematics, that a Book ſo worthy of its celebrated Author, does no longer remain in darkneſs.*

This Work would be much more uſeful, if ſome things in it, elegantly, tho' brief-

ly expreſſed, were more adapted to the Capacities of Beginners; and if others, which are barely touch'd on, were more clearly explain'd. Thoſe Writings that contain much Matter expreſſed in few Words, are indeed the moſt agreeable to Mathematicians, but not the moſt uſeful to all that apply themſelves to theſe Sciences.

This Defect might be corrected, and yet the Mathematicians have no cauſe to complain, if ſuch Writings were illuſtrated with Commentaries, which might be neglected by thoſe that found they did not need them.

Some of the chief Mathematicians of the preceding Age, have not thought Cartes's *Geometry unworthy their Comments. Neither is Sir* Iſaac Newton's *Arithmetic leſs deſerving. It is therefore to be wiſhed, that among ſo many eminent Mathematicians that now flouriſh, ſome one or other may undertake it.*

And that we may engage Mathematicians to reflect on the Neceſſity of ſuch a Commentary, we give this Specimen; wherein we endeavour to illuſtrate two Places, which are far from being the moſt difficult.

A Specimen of a Commentary on Sir Iſaac Newton's *Univerſal Arithmetic.*

Page 42.

Of the Invention of Diviſors.

TO this may be referred the Invention of Diviſors by which any Quantity may be divided.

1. *If it is a ſimple Quantity, divide it by its leaſt Diviſor, and the Quotient by its leaſt Diviſor, and ſo on till there remains an indiviſible Quotient. So you will have all the prime Diviſors of this compound Quantity; then multiply each two, three, four,* &c. *of theſe Diviſors together, and you will have alſo all the compound Diviſors.* As if all the Diviſors of 60 were required, divide it by 2, and the Quotient 30 by 2, and the Quotient 15 by 3, and there will be an indiviſible Quotient 5. Therefore the prime Diviſors are 1, 2, 2, 3, 5; thoſe produc'd by the Multiplication of each two are 4, 6, 10, 15; thoſe compos'd of three, 12, 20, 30.

and of all 60. Again, if all the Diviſors of the Quantity $21abb$ are deſir'd, divide it by 3, and the Quotient $7abb$ by 7, and the Quotient abb by a, and the Quotient bb by b, and there will be an indiviſible Quotient b; therefore the prime Diviſors are, 1, 3, 7, a, b, b; thoſe compounded of each two of theſe, 21, $3a$, $3b$, $7a$, $7b$, ab, bb. of three $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb; of four $21ab$, $21bb$, $3abb$, $7abb$; of five $21abb$. In the ſame manner all the Diviſors of $2abb-6aac$, are 1, 2, $bb-3ac$, $2a$, $2bb-6ac$, $abb-3aac$, $2abb-6aac$.

2. *If the Quantity, after it is divided by all its ſimple Diviſors, remains Compound, and there is a Suſpicion that it has a compound Diviſor, order it according to the Dimenſions of any Letter that is in it; and inſtead of that Letter ſubſtitute ſucceſſively three or more Terms of this Arithmetic Progreſſion,* $3, 2, 1, 0, -1, -2$; *and place the Numbers reſulting from ſuch a Subſtitution with all their Diviſors againſt its correſpondent Term in the Progreſſion, ſuppoſing the Signs of the Diviſors to be as well Affirmative as Negative. Then ſeek all the Arithmetic Progreſſions, that can be found amongſt theſe Diviſors, proceeding from the higheſt Terms to the loweſt, in the ſame order that the Terms of the Progreſſion* $3, 2, 1, 0, -1$, *proceed; and the difference of the Progreſſions thus ſought, muſt be either Unity,*

Unity, or ſome other Number that divides the higheſt Power of the Quantity propos'd: If any ſuch Progreſſion occurs, that Term of it in the ſame row with 0 *of the firſt Progreſſion, divided by the difference of the Terms, and then annex'd with its Sign to the Letter abovemention'd, is the compound Diviſor to be tried.*

3. As ſuppoſe the Quantity is $x^3-xx-10x+6$. Inſtead of x I ſubſtitute ſucceſſively the Terms of this Progreſſion 1, 0, −1; the Numbers by that means produc'd are −4, 6−14, which I place with all their Diviſors in the ſame Row with their correſponding Terms of the Progreſſion, 1, 0, −1 in this manner.

1	4	1, 2, 4,	+4
0	6	1, 2, 3, 6,	+3
−1	14	1, 2, 7, 14,	+2

Then becauſe the higheſt Term x^3 is diviſible by no Number but Unity, I ſeek a Progreſſion among the Diviſors, whoſe Terms differ by Unity, and which alſo decreaſes from above in the ſame manner with the Terms of the Progreſſion 1, 0 −1. I find but one Progreſſion in that manner 4, 3, 2, whoſe Term in the ſame row with 0 of the firſt Progreſſion 1, 0−1, is +3. I therefore annex this to x, and try the Diviſion by $x+3$, which ſucceeds the Quotient, being $xx-4x+2$.

4. Again let the Quantity be $6y^4 — y^3 — 21y^2 + 3y + 20$; I ſubſtitute inſtead of y 2, 1, 0, — 1, — 2 ſucceſſively, and place the Numbers reſulting 30, 7, 20, 3, 34, with all their Diviſors, in the ſame Row with the Terms that produc'd them as follows.

2	30	1, 2, 3, 5, 6, 10, 15, 30,	+10
1	7	1, 7,	+ 7
0	20	1, 2, 4, 5, 10, 20,	+ 4
—1	3	1, 3	+ 1
—2	34	1, 2, 17, 34.	+ 2

I can here find only this decreaſing Arithmetic Progreſſion +10, +7, +4, +1, —2; the common difference 3, divides the higheſt Term of the Quantity $6y^4$. Wherefore I annex the Term + 4 (which ſtands in the ſame Row with 0) divided by 3, the difference of the Terms to y, and try the Diviſion by $y+{}^{4}$; or, which is the ſame, by $3y+4$, this Diviſion ſucceeds the Quotient, being $2y^3 — 3yy — 3y + 5$.

5. And ſo in the Quantity $24a^5 — 50a^4 + 49a^3 — 140a^2 + 64a + 30$: the Operation will be as follows.

2	42	1. 2. 3. 6. 7. 14. 21. 42	+3	+3	+7
1	23	1. 2. 3.	+1	—1	+1
0	30	1. 2. 3. 5. 6. 10. 15. 30.	—1	—5	—5
—1	297	1. 3. 9. 11. 27. 33. 99. 297	—3	—9	—11

Here there are three Progreſſions, and their Terms —1 —5 —5, divided by the common differences 2. 4. 6, give the three Diviſors, $a-\frac{1}{2}$, $a-\frac{5}{4}$, $a-\frac{5}{6}$. By trying, I find that only the laſt ſucceeds, the Quotient being $4a^4-5a^3+4aa-20a-6$.

COMMENT.

6. In order to demonſtrate that our Author's Method neceſſarily diſcovers a Diviſor, as ſuppoſe $6a-5$, let us imagine it known. Now if in this Diviſor $6a-5$ we ſubſtitute any number for a, as 2; the number ariſing from thence will divide the number produc'd by ſubſtituting the ſame number 2 in the Quantity propos'd; for a may repreſent any number, and its Value is ſuppos'd to be the ſame in the Quantity and in its Diviſor.

By ſubſtituting inſtead of a in the Diviſor $6a-5$, the numbers 2. 1. 0. $-$1. $-$2. ſucceſſively, we ſhall have an Arithmetic Progreſſion, whoſe Terms are Diviſors of the Quantity propos'd, ſuppoſing the ſame Numbers, 2. 1. 0. $-$1. $-$2. to be put inſtead of a.

The difference of this Progreſſion is 6, which difference muſt be a Diviſor of 24, or elſe $6a$ would not divide $24a^5$, which it is neceſſary it ſhould, if $6a-5$ divides the Quantity propos'd: There muſt be given therefore amongſt the numeral Diviſors, an Arithmetic Progreſſion anſwering to the Diviſor $6a-5$, which muſt have thoſe Properties our Author requires.

The Term -5 anſwers to 0, becauſe it is had when 0 is ſubſtituted for a.

7. This Demonſtration may be applied to each of the Diviſors of the Quantity propos'd, and conſequently all are diſcover'd by this Method, becauſe each has a correſponding Progreſſion.

But as it is not certain that no Progreſſions can be diſcover'd, but thoſe that belong to each Diviſor; therefore he adviſes you to try each, and ſee if it divides the Quantity propos'd.

HE PROCEEDS.

8. If no Diviſor can be got this way, or none that divides the Quantity propos'd, we may conclude it has no Diviſor of one Dimenſion. But it may perhaps have one of two Dimenſions, if it is itſelf of more than three; and if ſo, that Diviſor may be ſought in this manner. *In that Quantity, inſtead of the Letter, ſubſtitute, as before, four or more Terms of this Progreſſion* 3, 2, 1, 0.—1.—2.—3. *and add and ſubſtract the Diviſors of all the reſulting Numbers, to and from the Squares of the correſponding Terms of the Progreſſion, multiplied into ſome numeral Diviſor of the Quantity propos'd, and place the Sums and Differences in a Row with that Term of the Progreſſion; then collect all the collateral Progreſſions that run through thoſe Sums and Differences. Let* $\mp$ C *be that Term of one of thoſe Progreſſions that ſtands oppoſite to* 0 *of the firſt Progreſſion,* $\mp$ B *the difference found by ſubſtracting* $\mp$ C *from its next ſuperiour Term that ſtands againſt* 1 *of the firſt Progreſſion,* A *the aforeſaid numeral Diviſor of the higheſt Term, and* l *the Let-*

ter in the Quantity propos'd; then $All \pm Bl \pm Cl$ *is the Diviſor to be tried.*

9. As if the Quantity propos'd was $x^4 - x^3 - 5xx + 12x - 6$; for x I write ſucceſſively 3, 2, 1. 0. —1. —2, and ſet the Numbers produc'd by that means 39. 6. 1 —6 —21 —26, together with all their Diviſors in order, and add and ſubſtract the Diviſors to and from the Squares of the correſponding Terms of the Progreſſion multiplied into a numeral Diviſor of the higheſt Term x^4 which is Unity; that is, to and from 9. 4. 1. 0. 1. 4; and place the Sums and Differences in the ſame row. Then I collect all the Progreſſions that can be form'd out of them as follows.

3	39	1.3.13.39	9	—30.—4.6.8.10.12.22.48.	—4.	6.
2	6	1.2. 3. 6	4	— 2.1.2.3.5.6.7.10.	—2.	3.
1	1	1.	1	0. 2.	0.	0.
0	6	1.2. 3. 6	0	— 6.—3.—2.—1.1.2.3.6.	2.	—3.
—1	21	1.3. 7.21	1	—20.—6.—2.0.2.4.8.22.	4.	—6.
—2	26	1.2.13.26	4	—22.—9.2.3.5.6.17.30.	6.	—9.

The Terms 2 and —3 of this Progreſſion which ſtand oppoſite to 0 in the firſt Progreſſion, I take ſucceſſively for $\mp$ C, and the difference of theſe and the Terms over them 0, 0, which is —2 and + 3, I take for $\mp$ B, Unity for A, and x for l; and ſo inſtead of $All \pm Bl + C$, I have two Diviſors to be tried $xx + 2x - 2$, and $xx - 3x + 3$; either of which ſucceeds.

10. Again, for the Quantity $3y^5-6y^4+y^3-8yy-14y+14$, the Operation will be as follows; firſt, I attempt the thing by adding and ſubſtracting the Diviſors to and from the Squares of the Progreſſion 2, 1, 0, 1, taking 1 for A, but it does not ſucceed; wherefore for A, I take 3, another numeral Diviſor of the higheſt Power $3y^5$, and then I add and ſubſtract the Diviſors to and from thoſe Squares multiplied by 3, that is, to and from 12. 3. 0. 3. and I find there are,

3	170		27		—7.	17
2	38	1.2.19.38.	12	—26.—71.0.11.13.14.31.50.	—7.	11
1	10	1.2. 5.10.	3	— 7.— 2. 1. 2. 4. 5. 8. 13.	—7.	5
0	14	1.2. 7.14.	0	14.— 7.— 2.—1.1.2.7.14.	—7.	— 1
—1	10	1.2. 5.10.	3	—7 — 2. 1. 2. 4. 5. 8. 13.	—7.	— 7
—2	190		12		—7.	—13

Theſe two Progreſſions amongſt the reſulting Terms, —7.—7.—7.—7. and 11. 5. —1.—7. for Expedition I neglect the Diviſors of the Extreams 170 and 190; wherefore continuing the Progreſſions, I take the Terms anſwering to them, *viz.* —7 and 17 above, and—7, and —13 below, and try if thoſe ſubducted from 27 and 12 in the fourth Column leave a difference that will divide 170 and 190; and I find the difference betwixt 27 and —7, that is 34, divides 170, and 19 the difference betwixt 12 and —7 divides 190; alſo the difference betwixt 27 and 17, *viz.* 10 divides 170, but 25 the difference of 12 and —13 does not di-

divide 190, wherefore I reject this laſt Progreſſion. Now in the firſt $\mp$ C is -7, and $\mp$ B nothing, the Terms of the Progreſſion having no difference; wherefore the Diviſor to be tried $All \pm Bl \pm C$ will be $3yy+7$, by which the Diviſion ſucceeds, the Quotient being $y^3-2yy-2y+2$.

COMMENT.

The Demonſtration of this Method differs very little from the Demonſtration of the forgoing one.

11. In the firſt Example (9) let $xx-3x+3$ be the Diviſor of the Quantity propos'd $x^4-x^3-5xx+12x-6$.

If this Diviſor is ſubſtracted from the Square of xx, the Remainder is $+3x-3$; if in this Quantity we ſubſtitute ſucceſſively 2, 1, 0. -1. -2. for x, we ſhall have an Arithmetic Progreſſion whoſe difference in aſcending is $+3$, and which when 0 is ſubſtituted, is -3, from which Progreſſion the Diviſor may be collected by the Rules of our Author.

Now ſubſtituting any Term of that Progreſſion, ſuppoſe 2, in the given Quantity, there will be among the Diviſors of the Number produced by ſuch a Subſtitution, a Number that will be found by putting 2 inſtead of x in the Diviſor $xx-3x+3$, and ſubtracting all the Diviſors from the Square of the Number ſubſtituted, *viz.* 4, there muſt be neceſſarily found amongſt the Differences ſo produc'd, that Term of the abovementioned Progreſſion, that anſwers to the Number 2.

From hence it follows, that this Progreſſion will neceſſarily be found amongſt thoſe Numbers, that the Author has ſought it in, and that every Diviſor will give a like Progreſſion.

Our Author does not only ſubſtract the numeral Diviſors from the Squares of the ſubſtituted Numbers; but he alſo adds them together, becauſe this Addition is the Subſtraction of a negative Diviſor.

Before the Addition or Subſtraction of the Diviſors, the Author multiplies the Square of the Number ſubſtituted by one of the numeral Diviſors of the higheſt Term, as in the ſecond Example by 3; the reaſon of which Operation will be manifeſt, if we apply the laſt Demonſtration to the Diviſor in the ſecond Example, $3yy+7$.

HE PROCEEDS.

12. If no Diviſor that ſucceeds can be found by theſe means, we may conclude that the Quantity propos'd does not admit of a Diviſor of two Dimenſions. The ſame method may be extended to the finding Diviſors of more Dimenſions, by ſeeking in the aforeſaid Sums or Differences, not Arithmetic Progreſſions, but ſuch whoſe firſt, ſecond or third Differences, are in Arithmetic Progreſſion: but the Learner need not be detained with this.

COMMENT.

13. It is not enough to diſcover ſuch Progreſſions, for it does not ſo eaſily appear how Diviſors can be form'd from ſuch Progreſſions. And beſides, Diviſors of three Dimenſions may be diſcover'd by Arithmetic Progreſſions only.

14. Let the Quantity to which we want Diviſors of three Dimenſions be

$$x^6 + 12x^3 + 4xx + 4x + 3$$

I form the firſt Column of nine or ten Terms of this Arithmetic Progreſſion.

4. 3. 2. 1. 0.—1.—2.—3.—4.

In the ſecond, I ſet down all the Numbers ariſing from the Subſtitution of each of theſe in the Quantity propos'd.

The third contains the Diviſors of the Numbers in the ſecond Column; all this is done the ſame in Diviſors of whatever Dimenſions.

The fourth Column is form'd of the Cubes of the reſpective Numbers in the firſt, multiplied by ſome numeral Diviſor of the higheſt Term of the Quantity propos'd with any number; that in the third Column ſtands oppoſite to 0, added to, or ſubſtracted from them.

Ex. gr. I multiply the Cubes 64. 27. 8. 1. 0.—1. —8.—27.—64. by one, the only numeral Diviſor of the higheſt Term x^6; and to the Products add, or ſubſtract from them, either 1 or 3, which ſtands in the third Column oppoſite to 0. I here add 3, and they become 67. 30. 11. 4. 3. 2.—5.—24.—61. In the ſame manner theſe three other Columns may be form'd.

61. 24. 5. —2 —3 —4 —11 —30 —67
65. 28. 9. 2 1 0 — 7 —26 —63
63. 26. 7. 0 —1 —2 —9 —28 —65

1st	2d	3d	4th	5th	6th	
4	4947		67		4.	
3	1104		30		2.	6
2	187	1. 11, 17, 187,	11	—88, —3, 0, 5, 6, 11, 14, 99,	0.	—3, 6
1	24	1. 2, 3, 4, 6, 8, 12, 24	4	—20, —8, —4, —2, 0, 1, 2, 3, 5, 6, 7, 8, 10, 12, 16, 28,	—2.	—2, 6
0	3	1. 3	3	0	—4.	—1, 6
—1	8	1. 2, 4, 8	2	—6—20, 1, 3, 4, 6, 10,	—6.	0, 6
—2	21	1. 3, 7, 21	—5	13, 6, 4, 3, 2, 1, —1, —8,	—8.	1, 6
—3	432		—24		—10.	2, 6
—4	3379		—61		—12.	

To each of theſe four ſeparate Columns, there muſt be the ſame Operation, applied, that we are now going to apply to the firſt of them.

I add to, and ſubſtract from each Number in the fourth Column, the correſponding Diviſors in the third Column, and ſet down the Sums or Differences (divided by the reſpective Numbers in the firſt Column) in the fifth Column.

I write no Number in the fifth Column oppoſite to 0, but conceive all Numbers Poſitive and Negative to be there.

It will be ſufficient to make uſe of only four or five of the leſſer Numbers of the ſecond Column, in order to form the third and fifth Columns, leaving the Places of the reſt vacant, as in our Example.

I now run over the Numbers of the fifth Column, to ſee what Arithmetic Progreſſion I can form out of them, and I diſcover five, which I ſet down in the *6*th Column: (I have here only ſet down three.) I then examine whether they can be continued upwards and downwards, by trying whether their Terms would have been found in their correſponding Lines of the fifth Column, if they had been continued, by which we avoid the long Computation that would have been neceſſary to continue the fifth Column.

15. Theſe Trials are only the reverſe of thoſe Operations that the fifth Column was form'd by. If the Progreſſion diſcover'd —2—4—*6*—8, is continued down, the Term of it that anſwers to —3 in the firſt Column, is —10, which Number ought to be in the fifth Column, that the Progreſſion may be continued; but if this is in the fifth Column, then if it is multiplied by —3, and the Product 30 ſubſtracted from its correſponding Term in the fourth Column, its

Remainder will be ſome Number in the 3d Column, that is, ſome Diviſor of the correſponding Number in the ſecond. Therefore if 6 exactly divides 432, the Progreſſion goes on; but if not, it is interrupted; and is therefore to be rejected.

It is ſufficient to try this Continuation in a few Terms, for otherwiſe the Operations would take up more time than the determining a Diviſor from the Progreſſion, and trying it; for every Progreſſion gives a Diviſor to be tried.

By trying their Continuation in this manner, all but the firſt Progreſſion is interrupted; and therefore this alone is to be made uſe of.

16. By the help of this Progreſſion, and the following Rules, I form the Diviſor to be tried.

1ſt. I multiply x^3 by the Diviſor of the firſt Term of the Quantity propos'd, by which I multiplied the Cubes of the Numbers of the firſt Column, in order to form the fourth. This Diviſor, in our Example, is 1, and the Product x^3 is the firſt Term of the Diviſor.

2ndly, The difference of the Progreſſion in aſcending, is $+2$; I change its Sign, and $-2x^2$ is the ſecond Term of the Diviſor.

3dly, The Term of the Progreſſion oppoſite to 0 in the firſt Column, is -4; I change its Sign, and $+4x$ is the third Term of the Diviſor.

4thly, Laſtly in forming the fourth Column, I added 3 to the Cubes of the firſt Numbers, and $+3$ is the laſt Term of the Diviſor.

The Diviſor to be tried is therefore

$$x^3-2xx+4x+3.$$

And trying it, I find it ſucceeds.

You muſt obſerve, that in the firſt and laſt Terms, the Signs are kept, and chang'd in the middle Terms.

DEMONSTRATION.

17. The Demonſtration of this Method has the ſame Foundation with the preceding one.

By ſubſtituting any Number, ſuppoſe 2, in the Diviſor of the Quantity propos'd, as in $x^3 - 2xx + 4x + 3$, inſtead of x there will be had a Number which exactly divides the Number produc'd, by ſubſtituting the ſame Number 2 in the Quantity propos'd. Such a Diviſor will therefore be found in the 3^d Column, anſwering to the Number ſubſtituted in the firſt.

The firſt Term of the Diviſor (it being of three Dimenſions) is x^3 multiplied by ſome numeral Diviſor of the firſt Term of the Quantity given.

The laſt Term of the Diviſor, is the Diviſor of the laſt Term of the Quantity propos'd, which is therefore to be found in the 3^d Column oppoſite to 0 in the firſt, if we ſuppoſe that negative Numbers may form this Column as well as affirmative ones.

Whence we may conclude, that the fourth Column contains the Sum of the firſt and laſt Terms of the Diviſor, when the correſponding Number in the firſt Column ſtands for x.

So our fourth Column is form'd by ſuch a Subſtitution in $x^3 + 3$.

And there can be no Diviſor, whoſe firſt Term and laſt together, does not form ſome fourth Column, the Number of which is ſometimes ſo great, that it makes the Method uſeleſs, for there are 192 if the firſt Term of the Quantity propos'd is multiplied by 24, and the laſt Term is 60.

If from this Sum $x^3 + 3$ we ſubſtract the Diviſor it ſelf $x^3 - 2xx + 4x + 3$, there remains the ſecond and third Terms, with their Signs changed $2xx-4x$, and dividing it by x, the Quotient is $2x-4$. Now from the Formation of the fifth Column, it is evident it muſt contain $2x-4$, ſuppoſing the correſpondent Number in the firſt Column ſubſtituted inſtead of x.

It is alſo evident, that by a ſucceſſive Subſtitution of the Numbers in the firſt Column for x; this Quantity $2x-4$, will form an arithmetic Progreſſion, whoſe difference is 2, and the Number oppoſite to 0 in the firſt Column is -4. It is alſo plain, that there can be no Diviſor of the Quantity propos'd, but what in ſome ſixth Column (of which every fourth will have one) will have an arithmetic Progreſſion by which the Diviſor is diſcover'd.

For the fourth which we make uſe of, gives the Sum of the firſt and laſt Terms, and the arithmetic Progreſſion diſcovers the two others.

HE PROCEEDS.

18. *When in the Quantity propos'd, there are two Letters, and all the Terms riſe to the ſame Number of Dimenſions, put Unity for one of thoſe Letters; then by the preceding Rules ſeek the Diviſor, and fill up the deficient Dimenſions of this Diviſor by reſtoriug that Letter for Unity.*

As if the Quantity was $6y^4-cy^3-21ccyy+3c^3y+20c^4$ where every Term is of four Dimenſions; I put 1 for c, and it becomes $6y^4-y^3-21yy+3y+20$, whoſe Diviſor is $3y+4$; and compleating the de-

deficient Dimenſion of the laſt Term by the Dimenſion of c, it becomes $3y+4c$, which is the Diviſor ſought. So if the Quantity is $x^4-bx^3-5bbxx+12bx-6b^4$, putting 1 for b, and finding the Diviſor of the remaining Quantity $x^4-x^3-5xx+12x-6$, which is x^2+2x-2; I compleat its deficient Dimenſions by the Diviſions of b, and ſo I have the Diviſor ſought, $xx+2bx-2bb$.

COMMENT.

19. It is is evident that the Quantity b muſt be in the Diviſor, and that it is only wanted where the Dimenſions are deficient; conſequently this wants no farther Demonſtration.

HE PROCEEDS.

20. When in the Quantity propos'd there are three or more Letters, and all its Terms have the ſame Number of Dimenſions, the Diviſor may be found by the preceding Rules.

COMMENT.

It will be evident by an Example how we muſt proceed, when there are three Letters in the Quantity propos'd.

21. Let the Quantity be,

$$\begin{aligned}12x^5-10ax^4-26aax^3+24a^3x^2-4a^3bx+12a^4b\\-9bx^4+12abx^3-8a^2bx^2+6a^2b^2x+32a^2b^3\\+6bbx^3-8ab^2x^2-12ab^3x-12b^5\\-24b^3x^2+18b^4x\end{aligned}$$

1st, For a and b, I ſubſtitute Unity, and the Quantity propos'd is chang'd into this,

$$12x^5 - 19x^4 - 8x^3 - 16x^2 + 8x + 32.$$

Whoſe Diviſor of two Dimenſions I diſcover to be

$$4xx - x - 4.$$

2ndly, For a I ſubſtitute 0, and Unity for b; and then the Quantity is chang'd into this,

$$12x^5 - 9x^4 + 6x^3 - 24xx + 18x - 12.$$

Whoſe Diviſor of two Dimenſions is

$$4xx - 3x + 2.$$

3dly, For a I ſubſtitute 1, and 0 for b, and I have

$$12x^5 - 10x^4 - 26x^3 + 24xx.$$

Which I divide by xx, and it becomes

$$12x^3 - 10x^2 - 26x + 24.$$

Whoſe Diviſor of two Dimenſions I find to be $4xx + 2x - 6$.

This is had by finding a Diviſor of one Dimenſion $3x - 4$, and dividing by it the Product is $4xx + 2x - 6$.

Theſe three Diviſors are the ſame Diviſor ſought.

$$4xx - x - 4.$$
$$4xx - 3x + 2.$$
$$4xx + 2x - 6.$$

And therefore that Term in which x or its Powers are alone, is found in each of them, that is

$$4xx.$$

In the ſecond Diviſor, all the Terms are deficient in which *a* is given, and all with *b* alone, or *b* and *x* together, will be had by compleating the Dimenſions of *b*; theſe Terms therefore are,

$$-3bx + 2bb.$$

In the ſame manner the third Diviſor gives the Terms that have *a*, or *a* and *x* in them.

$$+2ax - 6aa.$$

The Term of the Diviſor that has *ab* in it, is diſcover'd by collecting into one Sum the laſt Terms $+2$, -6 of the ſecond and third Diviſors; which Sum we ſubſtract from the laſt Term -4 of the firſt Diviſor, and multiply *ab* by the Remainder, keeping its Sign, in this Example the Remainder is 0, which ſhows that Term not to be in this Diviſor.

The reaſon of this laſt Operation, is, that in the firſt Diviſor -4, ſhows how often *aa*, *bb* and *ab*, are found together in the Diviſor ſought, becauſe *a* and *b* are repreſented by Unity; $+2$ ſhows how often *bb*, and -6 how often *aa* is contained in the ſame Diviſor; therefore ſubſtracting $+2 -6$ from -4, we have the Number of *ab*.

Collecting now all the Quantities diſcovered into one Sum, we have the Diviſor ſought,

$$\begin{aligned}4xx - 3bx + 2bb.\\ + 2ax - 6aa.\end{aligned}$$

22. When various Diviſors are diſcover'd in each Subſtitution, they only are to be compar'd together that have the ſame firſt Term, as is very evident.

But if there are many found in each Subſtitution, that have the ſame firſt Term, the Operation will be very uncertain.

This ſame Method may be applied to four, five, or more Letters.

23. As for Example, let the Letters be *a*, *b*, *c*, *d*, *e*, there muſt be ten Subſtitutions made, and that Letter alone muſt be left untouch'd, according to whoſe Dimenſions the Quantity is ordered; let this be *a*, the Subſtitutions then will be

1. $b=1$. & $c=d=e=0$.
2. $c=1$. & $b=d=e=0$.
3. $d=1$. & $b=c=e=0$.
4. $e=1$. & $b=c=d=0$.
5. $b=c=1$. & $d=e=0$.
6. $b=d=1$. & $c=e=0$.
7. $b=e=1$. & $c=d=0$.
8. $c=d=1$. & $b=e=0$.
9. $c=e=1$. & $b=d=0$.
10. $d=e=1$. & $b=c=0$.

Theſe ten Subſtitutions produce as many different Quantities, whoſe Diviſors are to be ſought; and thence the ſame Diviſor will be diſcover'd ten times, and then by comparing them together you will find the true one.

24. If the Quantity propos'd has many Diviſors with the ſame firſt Term, then it is to be order'd according to ſome other Letter.

He Proceeds.

25. But the Diviſor will be diſcovered more expeditiouſly in this manner.

Seek all the Diviſors of thoſe Terms in which a certain Letter is not found, and

and then the Diviſors of all thoſe Terms in which ſome other Letter is not found; and ſo in like manner the Diviſors of thoſe Terms in which the third, fourth and fifth Letters are not found, if there are ſo many Letters, and run through all the Letters in this manner; then oppoſite to each Letter, place the reſpective Diviſors; and ſee if in any Series of Diviſors, running through all the Letters, thoſe parts that involve only one Letter, are repeated as many times as the number of Letters, leſs by one in the Quantity propos'd: And the parts involving two, as many times repeated as the Number of Letters, leſs by two, &c. *If ſo, all thoſe parts once taken with their reſpective Signs, will be the Diviſor ſought.*

As if the Quantity propos'd is, $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$. The Diviſors of one Dimenſion of the Terms $8b^3 - 12bbc - 4bcc + 6c^3$, in which x is not, will be found by the former Rules to be $2b - 3c$, and $4b - 6c$, the Terms $12x^3 + 9cxx + 8ccx + 6c^3$, in which b is not, have only one Diviſor, $4x + 3c$; and the Diviſors of $12x^3 - 14bx^2 - 12bbx + 8b^3$, where c is not, are $2x - b$, $4x - 2b$. I place theſe Diviſors in a Row, with the Letters x, b, c, as follows,

$$\begin{array}{r|l} x & 2b - 3c.\ 4b - 6c. \\ b & 4x + 3c. \\ c & 2x - b.\ 4x - 2b. \end{array}$$

Now as there are three Letters, and each part of all the Diviſors involves but one Letter, thoſe parts ought to be twice repeated in the Series of the Diviſors; but the parts $4b$, $6c$, $2x$, b of the Diviſors $4b - 6c$, and $2x - b$, occur but once; that is, are found no where but in that Diviſor they are a part of: wherefore I neglect thoſe Diviſors. There remain then only three Diviſors, $2b - 3c$, $4x + 3c$ and $4x - 2b$; theſe three go through all the Letters, and each of their Parts $2b$, $3c$, $4x$, is twice repeated in them as it ought, and with the ſame Signs too, if the Signs of the Diviſor $2b - 3c$ is chang'd, and turn'd into $-2b + 3c$; for the Signs of every Diviſor may be chang'd. I take therefore all the Parts of theſe $2b - 3c$, $4x$ with their Signs, and the Sum $2b + 3c + 4x$, will be the Diviſor ſought. For if you divide the Quantity propos'd by it, there will come out $3xx - 2bx + 2cc - 4bb$.

Again, if the Quantity is $12x^5 - 10ax^4 - 9bx4 - 26aax^3 + 12abx^3 + 6bbx^3 + 24a^3xx - 8aabxx - 8abbxx - 24b^3xx - 4a^3bx + 6aabbx - 12ab^3x + 18b^4x + 12a^4b + 32aab^3 - 12b^5$; I place the Diviſors of thoſe Terms, in which x is not oppoſite to x, the Diviſors of thoſe without a oppoſite to a, and thoſe of the Terms without b, oppoſite to b, as here,

$$\begin{array}{l|l} x & b,\ 2b,\ 4b,\ aa+3bb,\ 2aa+6bb,\ 4aa+12bb, \\ & \quad bb-3aa,\ 2bb-6aa,\ 4bb-12aa, \\ a & 4xx-3bx+2bb,\ 12xx-9bx+6bb, \\ b & x,\ 2x,\ 3x,\ 4a,\ 6x-8a,\ 3xx-4ax,\ 6x^2. \\ & \quad 8ax,\ 2ax+ax,\ 3aa,\ 4xx+2ax,\ 6aa. \end{array}$$

Now I ſee that all of one dimenſion muſt be rejected, becauſe the ſimple ones b $2b$, $4b$, x $2x$, and the parts of the Compound ones $3x-4a$, $6x-8a$, occur but once in all the Diviſors; but there are three in the Quantity propos'd, and thoſe parts involve only one Letter, therefore they ought to be found twice. In like manner I reject the Diviſors $aa+3bb$, $2aa+6bb$, $4aa+12bb$, $bb-3aa$, and $4bb-12aa$ of two Dimenſions, becauſe their parts aa, $2aa$, $4aa$, bb, and $4bb$, involving only one Letter, are found only once; but the parts of the Diviſor $2bb-6aa$, which only remains in the Row with x, which alſo involve one Letter, are found again; that is, the part $2bb$ in the Diviſor $4xx-3$, $bx+2bb$, and the part $6aa$ in the Diviſor $4xx+2ax-6aa$: And theſe three Diviſors ſtand oppoſite to the three Letters x, a, b, and all their parts, $2bb$, $6aa$, $4xx$, which involve only one Letter, are found twice in them under their proper

Signs; but the parts $3bx$, $2ax$, which involve two Letters, occur only once. Wherefore the different Parts $2bb$, $6aa$, $4xx$, $3bx$, $2ax$ of thoſe Diviſors, connected under their proper Signs, makes the Diviſor deſired $2bb - 6aa + 4xx - 3bx + 2ax$. I therefore divide the Quantity propos'd by it, and there ariſes $3x^3 - 4axx - 2aab - 6b^3$.

COMMENTARY.

The Demonſtration of this Method is very eaſy.

26. Let the Diviſor of the Quantity propos'd be

$$\begin{array}{l} 4xx - 3bx + 2bb. \\ \quad\quad\;\; + 2ax - 6aa. \end{array}$$

The Author ſubſtitues o ſucceſſively for each Letter, rejecting every Quantity which has that Letter in it, both in the Diviſor and Dividend, by which means the remaining Diviſor divides the remaining Dividend as before.

This Diviſor, taking away x, becomes

$$2bb - 6aa.$$

which divides the Quantity propos'd, when x is taken out of it, that is, the Diviſor of that part of the Quantity propos'd, that has not a in it.

Taking away b, the Diviſor is

$$4xx + 2ax - 6aa.$$

Taking away a, it is

$$4xx - 3bx + 2bb.$$

Which Diviſors muſt be neceſſarily found amongſt thoſe Diviſors that the Author makes uſe of.

It is alſo evident, that there are as many Diviſors as Letters, that is, the ſame Diviſor ſo often repeated.

In which repetition the Quantities that contain only one Letter, will be neceſſarily in all; but where o is put for that Letter, the Quantities that contain two Letters, will be found every where, but in thoſe two where o is put for thoſe Letters.

Whence it is manifeſt, that the Author diſcovers by his method, all the Terms of the Diviſors ſought.

29. If there are many Diviſors of the ſame Dimenſions, the Series of the Diviſors that are run through, are to be ſeparated, and each will give one Diviſor.

What remains of the Author relating to this Affair is here added, but it does not need a Commentary.

HE PROCEEDS.

30. *If all the Terms of any Quantity are not equally high, the deficient Dimenſions are to be filled up by the Dimenſions of any aſſum'd Letter, and then when the Diviſor is found, that Letter is to be blotted out in both.* As if the Quantity was $12x^3 - 14bxx + 9xx - 12bbx - 6bx + 8x + 8b^3 - 12bb - 4b + 6$, aſſume any Letter, as c, and compleat the Dimenſions of the propos'd Quantity with it in this manner; $12x^3 - 14bx^2 + 9cxx$

$- 12\,bbx - 6\,bcx + 8\,ccx + 8\,l^3 - 12bbc - 4bcc + 6\,c^3$; then finding the Diviſor of this $4x - 2b + 3c$, take away c, and there will be the Diviſor deſir'd $4x - 2b + c$.

31. Sometimes Diviſors are more eaſily diſcover'd than by theſe Rules; as if ſome Letter in the Quantity propos'd is only of one Dimenſion, then you muſt ſeek the common Diviſor of thoſe Terms that have that Letter, and thoſe that have not, for that common Diviſor divides the whole; and if there is no common Diviſor of this ſort, then there will be no Diviſor of the whole: as if there was propos'd the Quantity $x^4 - 3ax^3 - 8\,aaxx + 18\,a^3x + cx^3 - acxx - 8aacx + 6\,a^3\,c - 8\,a^4$, let there be ſought the common Diviſor of the Terms $+ cx^3 - acxx - 8\,aacx + 6a^3c$, in which c is only of one Dimenſion, and of the remaining Terms $x^4 - 3\,ax^3 - 8\,aaxx + 18\,a^3x - 8\,a^4$, and that Diviſor $xx + 2\,ax - 2\,aa$ divides the whole Quantity.

Sir ISAAC NEWTON'S Univerſal Arithmetic,

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A RULE FOR EXTRACTING THE HIGHER ROOTS OF NUMERAL QUANTITES OF TWO PARTS COMMENSURABLE IN POWER.

Let the Quantity be $A+B$, *its greateſt part* A, *the Index of the Root to be extracted c; ſeek the leaſt Number n, whoſe Power* n^c, *can be divided by* A^2-B^2, *without a Remainder, and let the Quotient be* Q, *compute* $\sqrt[c]{\overline{A+B}\times\sqrt{Q}}$ *in the neareſt integer Numbers, and ſuppoſe it r, divide* $A\sqrt{Q}$ *by its greateſt rational Diviſor, let the Quote be s, and take* $\frac{r+\frac{n}{r}}{2s}$ *in the neareſt integer Numbers be t, and* $\frac{ts+\sqrt{ttss-n}}{\sqrt[2c]{Q}}$ *will be the Root ſought, if the Root can be extracted.*

33. As if the Cube Root was to be extracted from $\sqrt{968}+25$, $AA-BB$, will be 343 its Diviſors 7, 7, 7; therefore $n=7$, and $Q=1$. Again, $\overline{A+B}\times\sqrt{Q}$ or $\sqrt{968}+25$ (extracting the firſt part of the Root) is a little more than 56 its Cube Root in the neareſt Numbers is 4,

therefore $r = 4$, alſo $A\sqrt{Q}$ or $\sqrt{968}$, extracting all that is rational, becomes $22\sqrt{2}$; therefore its radical Part $\sqrt{2}$ is s and $\frac{r+\frac{r}{n}}{2s}$ or $\frac{5\frac{3}{4}}{2\sqrt{2}}$ in the neareſt integer Numbers is 2, conſequently $t = 2$. Laſtly, ts is $2\sqrt{2}$, $\sqrt{\overline{ttss-n}}$ is 1, and $\sqrt[2c]{Q}$ or $\sqrt[6]{1}$ is 1, therefore $2\sqrt{2}+1$, is the Root ſought, if the Root can be extracted. I try therefore by Multiplication, if the Cube of $2\sqrt{2}+1$ is $\sqrt{968}+25$, and I find it ſucceeds.

34. Again, if the Cube Root was to be extracted of $68-\sqrt{4374}$, $AA-BB$ will be 250, whoſe Diviſors are 5, 5, 5, 2; therefore $n = 5\times 2 = 10$, and $Q = 4$, and $\sqrt[6]{\overline{\overline{A+B}\times\sqrt{Q}}}$ or $\sqrt[3]{\overline{\overline{68+\sqrt{4374}}\times 2}}$, is in the neareſt integer Numbers, $7 = r$; alſo $A\sqrt{Q}$, or $68\sqrt{4}$, extracting the rational part, becomes $136\sqrt{1}$; therefore $s = 1$ and $\frac{r+\frac{n}{r}}{2s}$ or $\frac{7+\frac{10}{7}}{2}$ is in the neareſt integer Numbers $4 = t$: therefore $ts = 4$ $\sqrt{\overline{ttss-n}} = \sqrt{6}$ and $\sqrt[2c]{Q} = \sqrt[6]{4}$ or $\sqrt[3]{2}$; and therefore the Root to be tried is $\frac{4-\sqrt{6}.}{\sqrt[3]{2}}$

35. Again, if the Quadrato-Cubic of the fifth Root was to be extracted from

$29\sqrt{6}+41\sqrt{3}$ then $AA-BB=3$, and therefore $n=3$, $Q=81$, $r=5$, $s=\sqrt{6}$, $t=1$, $ts=\sqrt{6}$, $\sqrt{ttss-4}=\sqrt{3}$, & $\sqrt[2c]{Q}=\sqrt[10]{81}$ or $\sqrt[5]{9}$, and therefore the Root to be tried $\dfrac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$.

36. But if in theſe Operations the Quantity is a Fraction, or if its Parts have a common Diviſor, extract ſeparately the Roots of the Denominator and of the Factors: As if the Cube Root was to be extracted from $\sqrt{242}-12\frac{1}{2}$, this by reducing its Parts to a common Denominator becomes $\dfrac{\sqrt{968}-25}{2}$; then extract ſeparately the Cube Roots of the Numerator and Denominator, and there ariſes $\dfrac{2\sqrt{2}-1}{\sqrt[3]{2}}$.

37. Again, if any Root was to be extracted from $\sqrt[3]{3993}+\sqrt[6]{17578125}$, divide its Parts by their common Diviſor, $\sqrt[3]{3}$, and it becomes $11+\sqrt{125}$; whence the Quantity propos'd is equivalent to $\sqrt[3]{3}$ into $11+\sqrt{125}$, whoſe Root is found by extracting ſeparately the Roots of both the Factors $\sqrt[3]{3}$ & $11+\sqrt{125}$.

COMMENT.

This Method is eaſier in Practice than thoſe we find in other Authors, and more univerſal than what *Francis Van Schooten* has given, for extracting the Roots of Binomials, and which he has demonſtrated as to the Cube Root, in his Commentary on *Des Cartes*'s Geometry.

But this Method of Sir *Iſaac Newton*'s needs another Demonſtration, ſince *Schooten*'s can only be applied to one part of it.

LEMMA I.

38. *If the Binomial* $a \pm b$ *is raiſ'd to any Power whoſe Index is* c, *and if the Terms of this Power alternately taken, as the* 1st, 3d, 5th, 7th, &c. *are collected into one Sum, and ſo the whole Power divided into two Parts, the Difference of the Squares of the Parts will be* $\overline{aa-bb}^{\,c}$.

This will be manifeſt by the Analogy of the Operations, and it would therefore be unneceſſary to detain you with ſeeking a long univerſal Demonſtration.

The Square of the Root $a \pm b$ is

$$aa \pm 2ab + bb,$$

and the Parts are

$$\begin{aligned} &aa + bb \\ &\pm 2ab \end{aligned}$$

The Difference of their Squares is the Square of the Root $aa - bb$.

The Parts of the Cube of $a \pm b$ are

$$\begin{aligned} &a^3 + 3abb \\ &\pm 3a^2b \pm b^3. \end{aligned}$$

The Difference of their Squares is the Cube of the Root $aa - bb$.

This will appear by performing the Operations, and he that will continue them to the higher Powers, will eaſily perceive that the Propoſition muſt neceſſarily hold good in all.

LEMMA 2.

39. *If* a *and* b *are two Numbers, of which* a *is the greateſt, and the Binomial* $\sqrt{a} + \sqrt{b}$ *is raiſed to the Power* c, *and this is an odd Number; this Power will be a Binomial, one of whoſe Members is multiplied by* $\sqrt{a}$ *and the other by* $\sqrt{b}$; *and theſe two Members will be the Parts mentioned in Lem.* 1. *of which that is the greateſt which is multiplied by* $\sqrt{a}$.

As c is an odd Number, $c - 1$ is even, and may be divided into two Parts: I put therefore $2n = c - 1$.

Now if we raiſe the Binomial given to any Power whoſe Index is the odd Number c, we ſhall ſee that theſe Terms, neglecting their Signs and Coefficients, will be

$a^n\sqrt{a}$, $a^n\sqrt{b}$, $a^{n-1}b\sqrt{a}$, $a^{n-1}b\sqrt{b}$, $a^{n-2}b^2\sqrt{a}$, $a^{n-2}b\sqrt{b}$, &c.

And collecting into one Sum the 1, 3, 5, &c. Terms, we ſhall have a rational Number multiplied by $\sqrt{a}$; and the Sum of the 2, 4, 6, &c. Terms is a rational Number multiplied by $\sqrt{b}$.

And if we try this in any given Power, we ſhall ſee that theſe Sums conſiſt of correſponding Terms, which only differ in this, that where a is in the one, b is found in the other; wherefore as a is greateſt, that Term which has a in it when the other has b in it, will be greateſt; that is, the Term multiplied by $\sqrt{a}$ will exceed the other.

COROLLARY.

40. If either of the Members of the Roots, that is $\sqrt{a}$ or $\sqrt{b}$, is rational, that Member of the Power which is multiplied by the rational part, will be alſo rational; and the rational part of the Member Power will exceed the other, if the rational Term of the Root exceeds the other.

LEMMA 3.

41. *The ſame things ſuppos'd, if the Number* c *is an even Number, the Power forms a Binomial, one of whoſe Members is rational, and the other is multiplied by* $\sqrt{ab}$; *the Members are thoſe Parts of which we have treated in Lem.* 1.

For putting $2m = c$, (in which Caſe m is an Integer) and neglecting the Signs and Coefficients, the Terms are

$$\begin{array}{cccc} 1. & 3. & 5. & 7. \\ a^{m}, & a^{m-1}b, & a^{m-2}bb, & a^{m-3}b^{3}, \end{array}$$

$$\begin{array}{ccc} 2. & 4. & 6. \\ a^{m-1}\sqrt{ab}, & a^{m-2}b\sqrt{ab}, & a^{m-3}bb\sqrt{ab}; \end{array}$$

as will appear if the Operation is gone through, for any Power whoſe Index is an even Number.

COROLLARY.

42. If one of the Terms of the Root $\sqrt{a}$ or $\sqrt{b}$ is rational, the irrational Member of the Power is multiplied by the ſame irrational

Quantity that multiplies the rational Root: As ſuppoſe $a=ee$ the Term $\sqrt{a}$ will now be rational $=e$, and $\sqrt{ab}=e\sqrt{b}$, $\sqrt{b}$ is the irrational Quantity that multiplies the irrational Member of the Power.

LEMMA 4.

43. *Every Power of the numeral Binomial* $\sqrt{a}+\sqrt{b}$ *has both the Members poſitive, the Power of the Binomial or Apotome* $\sqrt{a}-\sqrt{b}$ *has one Member Negative; but the Members are the ſame whether it is* $+\sqrt{b}$ *or* $-\sqrt{b}$.

This follows from the Formation of Powers, as will be evident to any one that will try.

LEMMA 5.

44. *If the Binomial* $\sqrt{a}\pm\sqrt{b}$ *is raiſed to a Power whoſe Index is* c, *the Difference of the Squares of the Members of the Power, is* $\overline{a-b}^{c}$.

This follows from comparing Lem. 2 & 3, with Lem. 1.

LEMMA 6.

45. *The Root whoſe Index is* c, *that is* $\sqrt[c]{\ }$, *cannot be extracted from a Binomial, unleſs the Difference of the Squares of the Parts of the Binomial given has* $\sqrt[c]{\ }$ *of it rational.*

The Root, if it can be expreſſed, contains only ſquare Roots, as follows from *Lem.* 2 and 3. becauſe only ſuch Roots are contained in the Power.

Let us ſuppoſe now that the Root of the Binomial may be expreſſed, and let this be $\sqrt{a} \pm \sqrt{b}$, the Difference of the Square of the Parts of the Binomial given will be $\overline{a-b}^{\,c}$ (44); which when $\sqrt[c]{}$ is extacted is $a-b$, which is a rational Quantity: and this always happens when the Root can be expreſs'd.

LEMMA 7.

46. *If two continued decreaſing Geometric Progreſſions have a common middle Term, the Difference of the correſponding firſt Terms of the two Progreſſions will be greater than the Difference of the laſt Terms.*

Let the Progreſſions be

$$\div\!\!\div \; A,\ B,\ C$$
$$\div\!\!\div \; D,\ B,\ E$$

Therefore $A \times C = B \times B = D \times E$ and

$$A,\ D :: E,\ C$$

Divid. & Altern.

$$A-D,\ E-C :: D,\ C$$

But as the Progreſſions decreaſe, D exceeds B, which is greater than C; therefore D exceeds C, and A—D the Difference of the two firſt, exceeds E—C the Difference of the laſt.

LEMMA 8.

47. *The* $\sqrt[c]{}$ *cannot be extracted if* c *is an even Number, unleſs the greateſt Member of the given Binomial is rational.*

If $\sqrt[c]{}$ can be extracted, the Square Root can be extracted from the given Binomial; for it is $\sqrt[c]{}$ raiſed to the Power $\frac{1}{2}c$, which Index is an integer Number, as c is an even Number. Let the Square Root of this given Binomial be $a \pm b$, whoſe Square $aa \pm 2ab + bb$ is the Binomial propos'd.

Now the Root $a \pm b$, contains only ſquare Roots (39, 41); wherefore $aa + bb$ is the rational Member of the given Binomial, and $2ab$ the irrational Member.

We muſt demonſtrate therefore $aa + bb$, always exceeds $2\,ab$, for then it will be certain that the Root cannot be extracted but when the rational Member is greateſt.

a and b are not equal, for if they were, the Quantity propos'd would be a ſingle rational Quantity, or 0.

Let us ſuppoſe therefore a and b unequal, it is clear that $aa, ab : : ab, bb$; therefore $aa + bb$ exceeds $2\,ab$ by Prop. 25. lib. *Eucl.*

DEMONSTRATION OF THE AUTHOR'S METHOD.

48. The given Binomial is $A \pm B$; n and Q are found, ſo that $AAQ - BBQ = n^c$.

Our Author does not ſeek the Root of the Binomial given, but of $A\sqrt{Q} \pm B\sqrt{Q}$; and when he has found this, he divides it by $\sqrt[c]{}$ of $\sqrt{Q}$, that is by $\sqrt[2c]{Q}$, that he may have the Root of the Binomial given $A \pm B$.

By this Preparation the Binomial acquires thoſe Conditions, without which the Root cannot be expreſſed (45). Now n^c is the difference of the Squares of the Members, $A\sqrt{Q}$ and $B\sqrt{Q}$, from which if you extract $\sqrt[c]{\ }$ we muſt have a rational Number n.

Let us ſuppoſe $x\sqrt{y} \pm \sqrt{z}$, to be the Root of the Binomial requir'd $A\sqrt{Q} + B\sqrt{Q}$, and $x\sqrt{y}$ to be the greater part. As we do not here conſider Fractions, of which the Author treats ſeparately, x, y, z, will be integer Numbers; for as there are no Fractions in the Power given, there will be none in the Root.

The difference of the Squares of the Members of the Binomial $x\sqrt{y} \pm \sqrt{z}$, raiſ'd to the Power c, that is, the Difference of the Squares $A\sqrt{Q}$ and $B\sqrt{Q}$ is $\overline{xxy - z}^c$ (44), therefore

$$\overline{xxy - z}^c = \overline{AAQ - BBQ}^c = n^c$$

And

$$xxy - z = n.$$

From this Equation we deduce this decreaſing Proportion.

$$∺ x\sqrt{y} + \sqrt{z}, \sqrt{n}, x\sqrt{y} - \sqrt{z}.$$

This following is alſo a decreaſing Proportion.

$$∺ r, \sqrt{n}, \frac{n}{r}.$$

For we can demonſtrate that r exceeds n. In the firſt Proportion the difference betwixt the firſt and third Term is $2\sqrt{z}$, that is, not leſs than two. As the Proportion increaſes, the Difference betwixt the firſt and ſecond Term exceeds half the difference betwixt the firſt and third; that is, $x\sqrt{y} + \sqrt{z}$, and $\sqrt{n}$ differ by more than Unity: but r does not differ $\frac{1}{2}$ from $x\sqrt{y} + \sqrt{z}$, becauſe it is expreſſed in the neareſt integer Number (32. 43.); therefore r differs leſs than $\sqrt{n}$, from a Quantity that is greater than it, conſequently r neceſſarily exceeds $\sqrt{n}$.

The difference between r and the Root $x\sqrt{y} + \sqrt{z}$, as we have ſeen, is leſs than $\frac{1}{2}$, the difference then betwixt $x\sqrt{y} - \sqrt{z}$ and $\frac{n}{r}$ (46) is leſs than $\frac{1}{2}$.

Now in the two Proportions that we have conſider'd, if $x\sqrt{y} + \sqrt{z}$ exceeds r, then $\frac{n}{r}$ is greater than $x\sqrt{y} - \sqrt{z}$. If therefore we collect into one Sum $x\sqrt{y} + \sqrt{z}$, $x\sqrt{y} - \sqrt{z}$; and alſo r, $\frac{n}{r}$ the difference of thoſe Sums $2x\sqrt{y}$, $r + \frac{n}{r}$ will be alſo leſs than $\frac{1}{2}$. In this caſe alſo the difference will be diminiſhed, becauſe one will correct the other.

Therefore the Difference betwixt $\frac{r + \frac{n}{r}}{2}$ and $x\sqrt{y}$, the greateſt Member of the Root, is leſs than $\frac{1}{4}$.

This preſuppoſed, there are four Caſes to be examined; for $A\sqrt{Q}$ is either rational or ſurd, and c in each Caſe may be either even or odd.

49. CASE I. Let $A\sqrt{Q}$ be rational, and c odd.

In this Caſe $x\sqrt{y}$ the greater part of the Root is rational (40), it is therefore an integer Number; for as we have ſaid before, we have nothing to do here with Fractions, therefore $\frac{r + \frac{n}{r}}{2}$ is it ſelf the greateſt Member of the Root, for an Integer cannot differ by leſs than Unity; but this differs by leſs than $\frac{1}{4}$, from the greateſt Member.

In this Caſe too $s=1$, and therefore $\frac{r+\frac{n}{r}}{2} = \frac{r+\frac{n}{r}}{2s} = t = ts$, and the greateſt Member is truly diſcovered by the Author's Rule.

50. CASE 2. If $A\sqrt{Q}$ is irrational, and c an odd Number.

Now $x\sqrt{y}$ is a ſurd Number, and the Quantity $A\sqrt{Q}$, reduced to its leaſt Terms, will have the ſame Radical with $x\sqrt{y}$ (39). Our Author ſeeks this Radical, and calls it s, therefore $s=\sqrt{y}$.

We have ſeen that the difference betwixt $\frac{r+\frac{n}{r}}{2}$ and $x\sqrt{y}$ is not $\frac{1}{4}$, they will differ leſs if they are each divided by s or $\sqrt{y}$, therefore $\frac{r+\frac{n}{r}}{2s}$ and x differ leſs than $\frac{1}{4}$; and conſequently x is the neareſt integer Number to $\frac{r+\frac{n}{r}}{2s}$ that is $t=x$.

But $s=\sqrt{y}$, therefore $ts=x\sqrt{y}$, and the firſt Member of the Root is juſtly determined.

51. CASE 3. If $A\sqrt{Q}$ is rational, and c an odd Number.

In this Caſe, not as in Caſe 1. it is certain that $x\sqrt{y}$ may be rational (41); therefore this 3^d^ Caſe may be ſubdivided into two.

When $x\sqrt{y}$ is rational, the Demonſtration of Caſe 1. takes place, and the greateſt part of the Root is diſcover'd.

But if $x\sqrt{y}$ is a ſurd Quantity, this Method does not diſcover the true Root; for becauſe $A\sqrt{Q}$ is rational $s=1$ always, and not to $\sqrt{y}$, as it ought to be; as we have demonſtrated in Caſe the ſecond. See the 2^d^ and 3^d^ Cor. following.

52. CASE 4. When $A\sqrt{Q}$ is irrational, and c an even Number.

The Root ſought cannot be extracted, in this Caſe (47); and it would be to no purpoſe to apply our Author's Rule, or any other to ſuch a Quantity.

53. Now having found $x\sqrt{y}$, the greateſt Member of the Root $= ts$, we muſt demonſtrate $\sqrt{z} = \sqrt{ttss - n}$, that is, $z = ttss - n$.

We have

$$x\sqrt{y} = ts.$$
$$xxy = ttss.$$

But as we have ſeen above

$$xxy - z = n.$$

Therefore ſubſtracting the laſt Equation from the firſt,

$$xxy - xxy + z = z = ttss - n.$$

Which was to be demonſtrated.

It is evident, that the Members of the Root are to be join'd with the ſame Sign that the Members of the Quantity propos'd are join'd with (43).

The Author ſays, the Root thus diſcover'd muſt be tried, becauſe the Demonſtration ſuppoſes, that the Root can be expreſſed by $x\sqrt{y} \pm \sqrt{z}$, and does only take place when the Root can be extracted; but it does not in the leaſt follow from the Demonſtration, that it is always poſſible.

We have deduc'd the following Corollaries from what has been already demonſtrated, in order the more fully to illuſtrate our Author's Method.

COROLLARY I.

54. *When* c *is an odd Number, this Method always diſcovers the true Root, if it can be extracted,* (49, 50).

COROLLARY 2.

55. *When* c *is an even Number, and the Root can be extracted, you will then diſcover the true Root, if either of the Members are rational.*

This follows from what has been already demonſtrated, if the greateſt Member of the Root is rational (51); but if this Member is irrational, the Root may be diſcover'd by making uſe of $B\surd Q$, inſtead of $A\surd Q$, to find s. The Reaſon of this will be evident, if you apply what has been demonſtrated to theſe Caſes; for it is certain that even here $s = \surd y$ (42).

Whence we deduce this Obſervation: As we cannot ſee before we begin the Operation, whether the greateſt or leaſt Member of the Root is rational; if the Root diſcovered by the Author, which has the greateſt Member rational, is not the true one, we muſt ſeek another, in which the greateſt Member may be irrational.

COROLLARY 3.

56. *If neither of the two Roots mentioned in the laſt Corollary is the true one, we muſt not from thence conclude that the Root cannot be extracted.*

For if both Members of the Root ſhould be irrational, yet $A\surd Q$ will be rational (47), and $s = 1$ (49) by our Author's Method. If according to the Obſervation in the laſt Corollary, we make uſe of B $\surd$ Q, then it will be $s = \surd yz$ (41); and we cannot deny but in this caſe our Author's Method fails; but this Defect will not much diminiſh its Uſe, if as *Van Schooten* has taught, in his Method, we firſt extract the Square Root, of the Quantity by the uſual Method; and by that means reduce the Problem to the Extraction of a Root, whoſe Index is an odd Number.

COROLLARY 4.

57. *When the Index of the Root is an even Number, and the greateſt Member of the Binomial propos'd is irrational, the Root cannot be expreſſed*; and it would be in vain to ſeek it, as we have already obſerv'd (52).

FINIS.

A New
METHOD
OF
Determining the FORM
OF AN
Aſſum'd Infinite Series.

THE

PREFACE.

WHEN a Quantity is ſought, that cannot be exactly expreſſed, its Value is to be inveſtigated by Approximation; and this Value, as to Practice, may be taken for the true Value.

No Mathematician can be ignorant of the Uſe of Infinite Series in theſe Approximations.

Amongſt the various Methods by which the Value of any Quantity, in any given Equation, may be expreſſed, by a converging Series; that is deſervedly preferred in many occaſions where an indetermin'd Series is aſſum'd, which is put equal to the Quantity propos'd; and then as many Terms of it are determin'd, as ſhall be thought proper.

Many things relating to theſe Solutions are to be met with in different

Authors; but they ſeem to be very little ſollicitous about explaining the Method itſelf. Thoſe who propoſe to explain it by Examples, ſuppoſe that known, in which the whole difficulty of the Solution conſiſts.

As for Example; let y *be to be determined from* x *known in any given Equation.*

They expreſs the Value of y *by an aſſum'd indetermin'd Series, putting*

$$y = Ax^{n} + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r}, \&c.$$

Then they explain how the Co-efficients A, B, C, D, *&c. are to be determin'd, in which indeed there is no difficulty. And Values are placed for* n *and* r, *as if they were determin'd with the greateſt eaſe, when indeed all the difficulty is in diſcovering theſe.*

Reyneau *has obſerv'd many Things relating to theſe Numbers,* Anal. demontrée Art. 246. *but does not in the leaſt explain by what means they are to be determin'd.*

Sir Iſaac Newton, *to whom Mathematics are ſo much indebted, has demonſtrated the Determination of the Number* n, *if we here apply what he has ſaid of the inveſtigating the firſt Term in his Explication of another method of Infinite Series.*

Mr. Taylor, *in his Book,* de Methodo Incrementorum, Prop. 9. *has treated of the Inveſtigation of the Number* r. *But Mr.* Sterling *has obſerv'd,* de Lineis tertii Ordinis Newton. pag. 28. *that the Rule he there lays down, does ſometimes fail; on which occaſion he himſelf gives another,* which, *he confeſſes,* he found by chance, and had ſought its Demonſtration in vain, wherefore he could not aſſert it was always true.

What made him doubt it, I cannot ſee; but tho' I am fully perſuaded that it will never fail, yet I cannot believe it abſolutely compleat, and will now ſhew wherein I take it to be imperfect.

The Computation may proceed when n *is diſcover'd, tho' the Form of the aſſum'd Series ſhould not be the ſame with the Form of the Series ſought, provided there is ſome Diviſor of* r *made uſe of inſtead of* r; *that is, we ſhall always have the ſame Series at laſt, if the aſſum'd Series is contain'd in the true Series.*

Example, let $n = 3$, *and* $r = 4$, *the Form of the Series is*

$$y = Ax^3 + Bx^7 + Cx^{10} + Dx^{13}, \&c.$$

Which Series being aſſum'd, we may go on with our Computation, but theſe alſo may be made uſe of

$$y = ax^3 + bx^5 + cx^7 + dx^9 + ex^{11} +, \&c.$$

$$y = ax^3 + bx^4 + cx^5 + dx^6 + ex^7 +, \&c.$$

Each of which contains the firſt: if now we proceed with our Computation, according to the given Equation, we ſhall find in the ſecond Series.

$$a = A,\ b = 0,\ c = B,\ d = 0,\ e = c,\ \&c.$$

In the third we ſhall have

$$a = A,\ b = 0,\ c = 0,\ d = 0,\ e = B,\ \&c.$$

So that in theſe three Subſtitutions we come at laſt to the ſame Series which has the Form of the firſt of the three.

By Mr. Sterling'*s Rule, we very often come not at the true Form, but at a Series that contains the true Form; ſo that there is ſome ſuperfluous labour to find the true one, which in ſome meaſure may be avoided, by ſeeking the true Form from one that contains it, before we begin the Computation, which how to do, we would give a particular Rule, if it was not leſs labour in Practice, to ſeek directly the true Form of the Series ſought.*

As to Mr. Taylor'*s Method, it muſt be obſerv'd, that* Sterling *himſelf, makes uſe of it, only correcting it in thoſe places where he thinks it fails; where we may remark, that this Correction is not always neceſſary when it is made uſe of.*

But notwithſtanding Mr. Taylor'*s* Method, *corrected by* Sterling, *is ſufficient in all caſes; I believe it would not be un-*

unacceptable to Beginners, to be ſhewn a Way by which they may avoid much trouble; for our Rule gives the true Value of r, *by which means the aſſum'd Series, immediately acquires the Form of the Series ſought. But that this may be the more uſeful, I will firſt ſet down the Determination of the Letter* n, *according to Sir* Iſaac Newton'*s Method, which Mr.* Taylor *explains in the forementioned 9th Propoſition; and Mr.* Sterling *in Prop. 2. of his Tract, where they apply it to aſſum'd indetermin'd Series.*

I ſhall not treat of determining the Coefficients A, B, C, D, *&c. becauſe this is done by a well-known Method made uſe of in many other occaſions.*

A NEW METHOD OF DETERMINING THE FORM OF AN ASSUM'D INFINITE INDETERMIN'D SERIES.

PROPOSITION I.

The Inveſtigation of the Index of the firſt Term of the Series.

LEt there be given an Equation, in which y is ſought, and its Value is to be expreſſed by the Powers of x.

1. Put y equal to an Aſſum'd Infinite Indetermin'd Series,

$y = Ax^{n} + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r} + Ex^{n+4r} +$ *&c.*

The Number n, is to be determin'd.

2. Let there be form'd a Parallelogram, repreſented in TAB. IV. and mark its Angles, as you there ſee; the Method of continuing it, or making it larger, is plain.

3. In the Equation propos'd, ſubſtitute x^{n} for y, and if there are Fluxions, for the firſt Fluxion of y, put x^{n-1}, for the ſecond x^{n-2}; and ſo of the reſt.

4. Mark in the Parallelogram all the Angles that contain Indices of the Powers of x, in the given Equation.

If it contains fractional Indices, the Points to be mark'd may be diſcover'd; $n + \frac{1}{2}$ muſt be mark'd in the middle, betwixt n and $n + 1$, and $\frac{3}{2}n$, is plac'd in the middle betwixt n and $2n$.

5. Draw a Line in the Parallelogram, that may paſs through two or three of thoſe Points that are mark'd, in ſuch a manner, that all the other Points that are mark'd, may fall on the ſame ſide of that Line; but you muſt obſerve, that this Line muſt not be drawn directly up and down, as AB or CD, *&c.*

6. Make all the Indices, through which the Line paſſes, equal amongſt themſelves, and n will by that means be determin'd.

EXAMPLE.

7. Let the Equation given be

$$x^3y^7 - a^5\frac{y^6}{x} - 6ba^2x^3y^4 - d^7xy^2 - a^8x^2 = 0.$$

For y ſubſtitute x^n (39), and the Indices are

$$7n+3,\ 6n-1,\ 4n+3,\ 4n+2,\ 2n+1,\ 2.$$

Theſe being mark'd in the Parallelogram, the Line may be drawn four ways (5);

1. Thro' 2, $2n+1$, $6n-1$, & $n = \frac{1}{2}$. (6).
2. Thro' $6n-1$, $7n+3$, & $n = -4$. (6).
3. Thro' $7n+3$, $4n+3$, & $n = 0$. (6).
4. Thro' $4n+3$, 2, & $n = -\frac{1}{4}$ (6).

8. If inſtead of n, in each Index, its Value is ſubſtituted, each of thoſe, through which the Line paſſes, which are equal to each other, will form an Index, either the greateſt of all, as in the two firſt Caſes; or the leaſt of all, as in the two laſt.

9. When this Index is the greateſt of all, all the Points mark'd in the Parallelogram, are found below the Line that is drawn, and the Series converges ſo much the faſter, as x is greater.

10. The contrary holds, when the Index is the leaſt of all.

PROPOSITION II.

The Determination of the Difference of the Indexes of two Terms, immediately following each other, in the aſſum'd Series.

11. In the aſſum'd Series abovementioned (1).

$$y = Ax^{n} + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r} + \&c.$$

r here repreſents the Difference that this Propoſition is to determine.

12. Seek n by the precedent Propoſition, and then in the Indexes of the Powers of x after the Subſtitution mentioned in n 3, inſtead of n put its Value.

13. Thoſe Indexes that are touch'd in the Parallelogram, and which are equal to each other (5), are to be taken for one: Subſtract this from all the reſt, if it's leſs; or if it's greater, ſubſtract them from it; you will form a Series which we call, *a Series of Differences.*

14. Seek the greateſt common Diviſor of theſe Differences.

15. By this divide the leaſt of thoſe Differences, and retain the Quotient.

16. Seek the Value of A, which when n is given, may always be done; and note the Number of equal Values, if there are not various equal Values, Unity will repreſent their Number.

If A has various different Values, let that determine the Number of equal Values, which is made uſe of in the Computation. Example, if a, a, a, b, b, c, are ſix Values of A, the number of equal Values will be 3, if we make uſe of a; 2, if we make uſe of b; and 1, if we make uſe of c.

17.

17. Seek the leaſt Number that can be exactly divided by the Number of equal Values of A, and the Quotient mentioned in n. 15.

18. Now divide the leaſt in the Series of Differences (13) by this Number, and the Quotient will be the Number ſought, that is r, which will be negative if the Series converges the faſter, as x is greater (9), and Affirmative in the other caſe mentioned in n 10.

19. Mr. *Taylor* ſeeks the greateſt common Diviſor of the Indexes; and this *Sterling* divides by the Number of equal Values of A.

This our Rule, may appear perhaps more complex; it will nevertheleſs diminiſh the trouble, and that it may be better comprehended, we will illuſtrate it with Examples.

EXAMPLE 1.

Let the given Equation be

$$x^5 - 4y^{\frac{1}{2}}x^4 + 6yx^3 - 4y^{\frac{1}{2}}xx + y^2x + 10x^{\frac{1}{3}} - 6y^{\frac{1}{2}} = 0.$$

Let y be ſought in a Series converging faſter as x is greater.

I put

$$y = Ax^n + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r} + \&c.$$

I ſubſtitute x^n in the Equation, inſtead of y (3.) and the Indexes are 5, $\frac{1}{2}n+4$, $n+3$, $\frac{3}{2}n+2$, $2n+1$, $\frac{1}{3}$, $\frac{1}{2}n$, and finding $n=2$, they are changed into theſe (12).

$$5,\ 5,\ 5,\ 5,\ 5,\ \tfrac{1}{3},\ 1$$

And 5 is the greateſt Index the Line touches, becauſe we here want the greateſt (9).

I ſubſtract the other Indexes from 5, and I have a Series of Differences (13).

$$4,\ \tfrac{2}{3},\ 4.$$

The greateſt common Diviſor of theſe is $\frac{2}{3}$ 14.)

By this I divide 4, the leaſt of the Differences (15.) and the Quotient is 6.

I ſeek A (16), and find it has four Values.

I ſeek the leaſt Number that can be exactly divided by 6, and 4 (17.) which is 12.

By this I divide 4, the leaſt of the Differences and the Quotient is $\frac{1}{3}$ and $r=-\frac{1}{3}$ (18).

The Form of the Series is therefore,

$$y=Ax^2+Bx^{\frac{5}{3}}+Cx^{\frac{4}{3}}+Dx+Ex^{\frac{2}{3}}+Fx^{\frac{1}{3}}+G$$
$$+Hx^{-\frac{1}{3}}\ \&c.$$

EXAMPLE II.

$$-\frac{y^9}{a^5}+xy^3-2x^2y^2+x^3y+\frac{x^{14}}{b^{10}}=0$$

Let y be ſought in a Series converging faſter as x decreaſes.

The aſſum'd Series is

$y=Ax^n+Bx^{n+r}+Cx^{n+2r}+Dx^{n+3r}+$ &c.

Inſtead of y in the Equation, I write x^n, (3.) and the Indexes of x are,

$9n$, $3n+1$, $2n+2$, $n+3$, 14.

Now in the Parallelogram, we find by Prop. I. $n=1$, and the Indexes are (12).

9, 4, 4, 4, 14.

The Series of Differences (13), is,

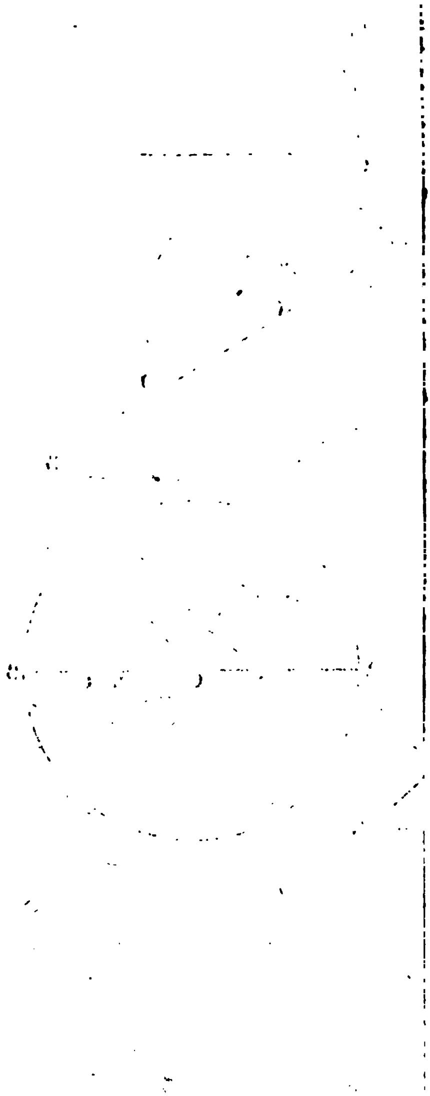
5, 10.

And 5 their greateſt common Diviſor, (14).

By which dividing the leaſt Difference, the Quotient is 1, (15).

A has two equal Values (16); and 2 is the leaſt Number that can be exactly divided by this and by the Quotient 1. (17).

I



I divide therefore 5 by 2, and $r = 2\frac{1}{2}$ (18). Then the Form of the Series is,

$$y = Ax + Bx^{3\frac{1}{2}} + Cx^{6} + Dx^{8\frac{1}{2}} + \&c.$$

And the Series it ſelf, is

$$y = x + \frac{\frac{2}{5}}{a^{2}}x^{3\frac{1}{2}} + \frac{4}{a^{5}}x^{6} + \left.\begin{matrix} \dfrac{22}{\overset{15}{a^{2}}} \\ -\dfrac{a^{\frac{5}{2}}}{2b^{10}} \end{matrix}\right\} x^{8\frac{1}{2}} \left.\begin{matrix} + \dfrac{140}{a^{10}} \\ - \dfrac{13}{2b^{10}} \end{matrix}\right\} x^{11} + \&c.$$

FINIS.